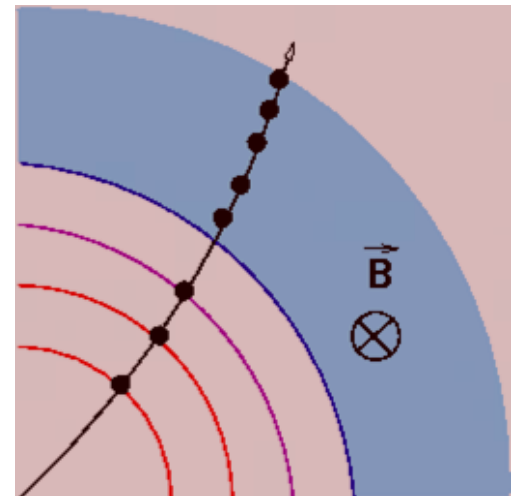
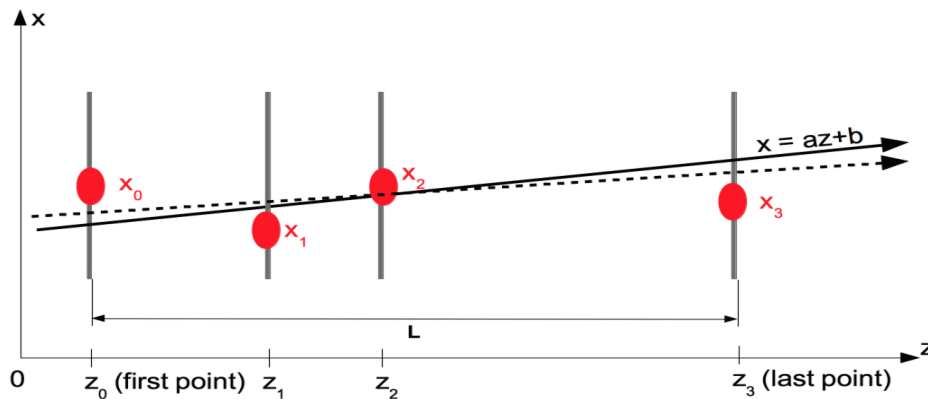


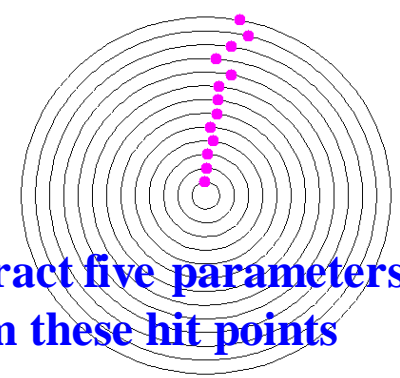
Introduction to Track Fitting

- How many parameters are required to define a charged trajectory in magnetic field ?
- In 2D, how many free parameters to define a straight line ?
- How many are in 3D ?
 - $(x,y,z) + (\cos\alpha, \cos\beta, \cos\gamma) : 6$
 - $(x,y,z) + (\theta, \phi) : 5$
 - In a plane/surface (One constraint) \rightarrow Four free parameters
- Trajectory in magnetic field : Four + one for the curvature : Total 5

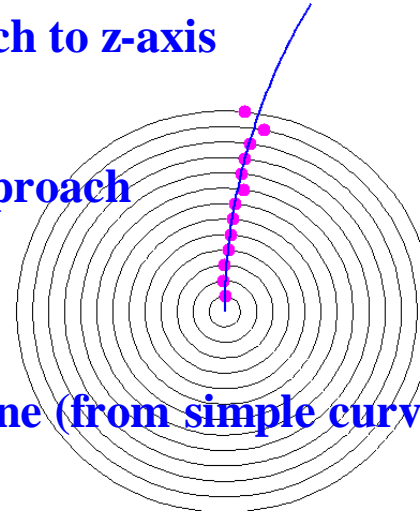


Track fitting : Introduction

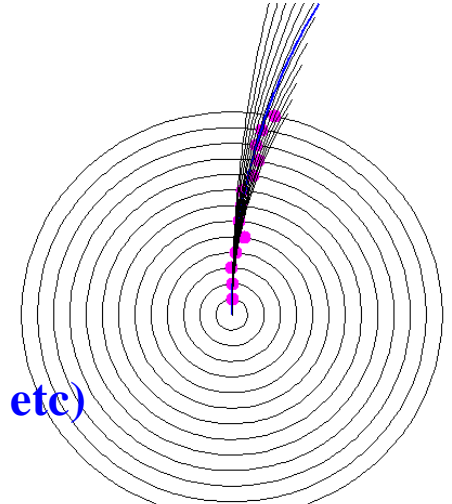
- These hit points belongs to a track (track finder) is given
 - Required to estimate the all track parameters \mathbf{p}
 $((\rho, \Phi_0)/(x, y)/(d_0, z_0), \phi, \cot(\theta), q/p_T)$ or $(x, y, \frac{dx}{dz}, \frac{dy}{dz}, \frac{q}{p})$
 - d_0 : signed distance of closest approach to z-axis
 - z_0 : z of signed closest approach
 - ϕ : Azimuthal angle at the closest approach
 - θ : Polar Angle of track
 - q/p_T : Charge-signed curvature



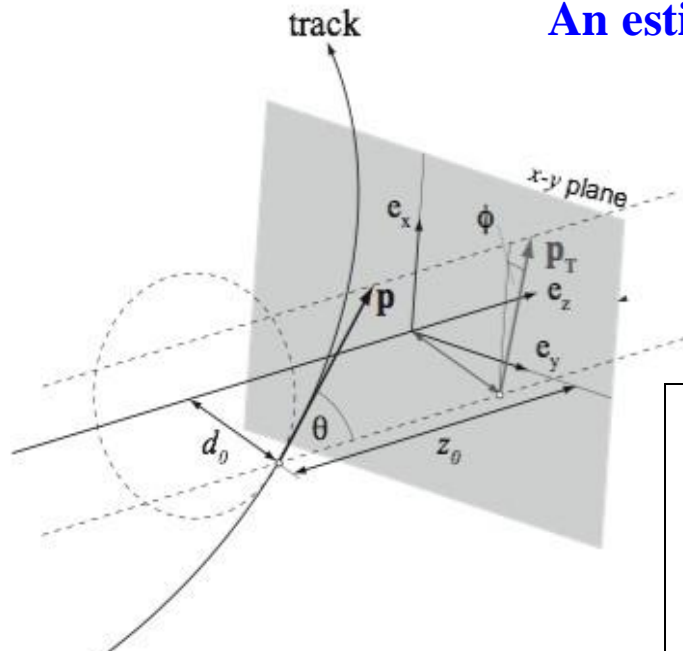
Extract five parameters from these hit points



An estimated one (from simple curvature etc)



Based on starting parameters, extrapolated points will vary (a function of \mathbf{p} , $\mathbf{f}(\mathbf{p})$). Minimise $\chi^2 = \sum_{x,y} \sum_i (m_i - f(\mathbf{p})_i)^2$ to obtain track parameters \mathbf{p} . Takes care of any correlation between measured points



- Analytical eqn of helical path can not be used directly
 - Magnetic field is not uniform
 - $-dE/dx$ of charge particle, change in curvature
- Use numerical approach

Track Fitting : Introduction

- **The aims of track fitting :**
 - **Compute the best estimate of the track parameter**
 - **Compute the covariance matrix of the estimate**
 - **Compute test statistics**

- **There are several types of mistakes :**
 - **Misses :** A hit is missing from a track candidate
 - **Contamination :** A track candidate contains a noise or a background hit
 - **Loss :** There is no candidate for a track
 - **Ghost :** A candidate is generated that does not correspond to track

Few remarks on track model

- The eqn of motion can be solved with sufficient precision
- Required a precise measurement of magnetic field and should be easily accessible
- Known material for estimation of multiple scattering, energy loss etc
- Removal of noise hits etc, wrong measurements
- Fit should have
 - Bias free, **Consistency**, **Robustness**
- Test for goodness of fit
 - Pull quantities, for correct estimations of track parameters these three condition must be fulfilled
 - The track model is correct
 - Covariance matrix of the measurement must be correct
 - Reconstruction programme must work properly
 - χ^2 test
- What are those track parameters ?
 - $\{ (\varphi_0, z)/(\rho, \varphi_0)/(x, y), \varphi, \cot(\theta), q/p_T \}$ or $\{ x, y, dx/dz, dy/dz, q/p \}$
 - Continuity with respect to small changes
 - Errors should be close to Gaussian (**1/p, not p**)
 - Good linear approximation to track propagation (reduces the effect of second derivatives)

Least Square fitting

- **Simpson approximation** : $f(p) = f(p_0) + A \cdot (p - p_0) + O(p - p_0)^2$
- **Minimisation of χ^2 gives the updated value of tracking parameter, p**

$$\chi^2 = [f(p_0) + A \cdot (p - p_0) - m]^T \cdot V^{-1} \cdot [f(p_0) + A \cdot (p - p_0) - m]$$

$$\Rightarrow p = p_0 + [A^T \cdot V^{-1} \cdot A]^{-1} \cdot A^T \cdot V^{-1} \cdot (m - f(p_0)) \quad \text{[derive this]}$$
- **Where**
 - $m = m_i$: vector of measurements, e.g., z & ϕ ($2 \times N$, number of track points)
 - $f = f_i$: vector of function corresponding to p, e.g., $(\phi_0, dz, \cot\theta, \phi, q/p_T)$
 - V : the covariance matrix of m, e.g., inverse error matrices, (both position and effect of multiple scattering)
 - p_0 : the approximate initial value of track parameters
 - $A = \partial f / \partial p$: at the point p_0 **[in general no analytic formula]**
- **The solution of the least-square problem, $p - p_0 = [A^T \cdot V^{-1} \cdot A]^{-1} \cdot A^T \cdot V^{-1} \cdot (m - f(p_0))$**

- **Covariance matrix**

$$V = \langle (p - p_0)(p - p_0)^T \rangle = \left\langle \left[[A^T V^{-1} A]^{-1} A^T V^{-1} \epsilon \right] \left[\epsilon^T V^{-1} A [A^T V^{-1} A]^{-1} \right] \right\rangle =$$

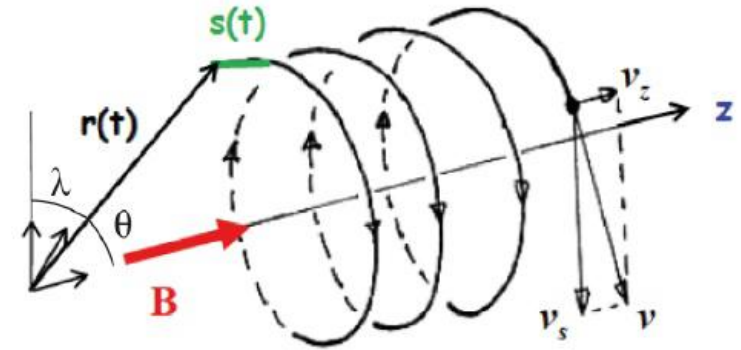
$$\left[[A^T V^{-1} A]^{-1} A^T V^{-1} V V^{-1} A [A^T V^{-1} A]^{-1} \right] = \left[[A^T V^{-1} A]^{-1} A^T V^{-1} V^{-1} A [A^T V^{-1} A]^{-1} \right] =$$

$$(A^T V^{-1} A)^{-1}, \quad \text{where } \epsilon = (m - f(p_0)) \text{ and } V = \langle \epsilon \epsilon^T \rangle$$

Extrapolation of track in presence of magnetic field

- For collider experiment in barrel part, where magnetic field is uniform and in Z-direction, track parameters after a pathlength along the helix, s is

- $p_x = p_{0x} \cos(\rho s) - p_{0y} \sin(\rho s)$
- $p_y = p_{0y} \cos(\rho s) + p_{0x} \sin(\rho s)$
- $p_z = p_{0z}$
- $x = x_0 + \frac{p_{0x}}{a} \sin(\rho s) - \frac{p_{0y}}{a} [1 - \cos(\rho s)]$
- $y = y_0 + \frac{p_{0y}}{a} \sin(\rho s) + \frac{p_{0x}}{a} [1 - \cos(\rho s)]$
- $z = z_0 + \frac{p_{0z}}{a} s$

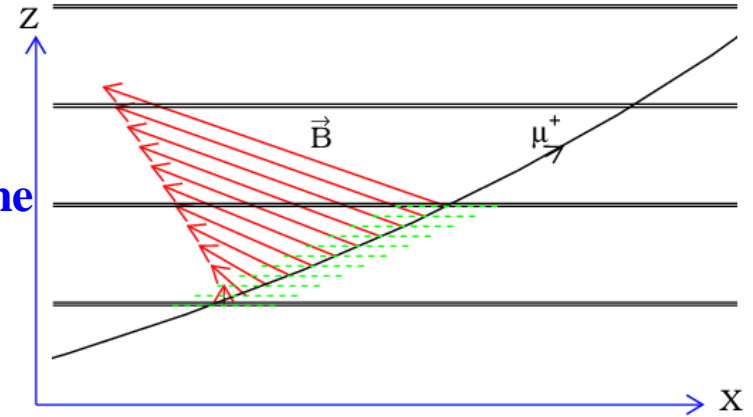


[Derived these expressions from $\vec{F} = q(\vec{v} \times \vec{B})$]

- Where $a = -(0.29979458) Bq$, and $\rho = a/p = a \sin\theta/p_T$
- Do not use the simple pathlength, s , extract it from the intersection of helix and cylinder [Calculate s from above equations after consider (x,y) lying on the surface of a cylinder of radius R]

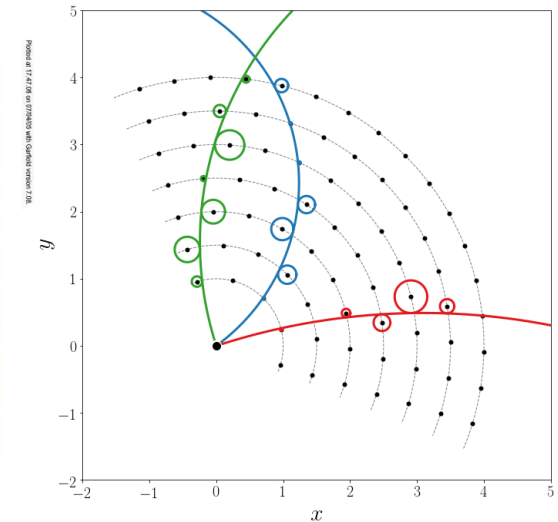
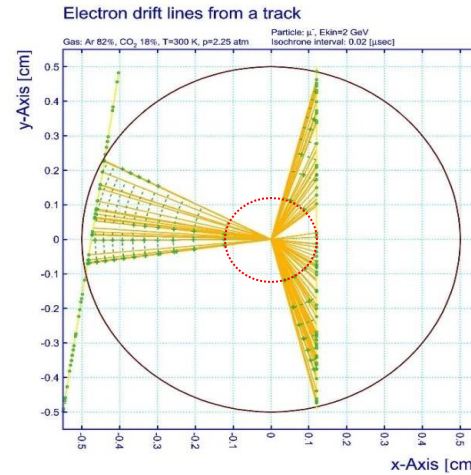
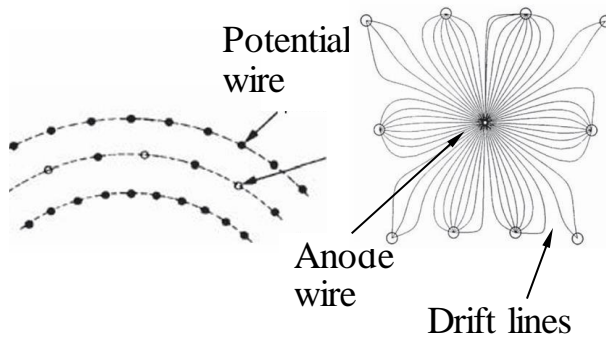
Extrapolation of track in presence of magnetic field

- No simple or direct propagation in case of inhomogeneous magnetic field
 - Track propagated through any standard package, e.g., Runge-Kutta method or
 - Assume uniform magnetic field locally
- Transform co-ordinate system such that magnetic field is along Z' axis
- Get distance to the crossing point of helix and plane
- Get the track parameters at the crossing point
- Return back to lab frame
- Step size is \sim mm, need optimisation of CPU time and performance



Some more remarks

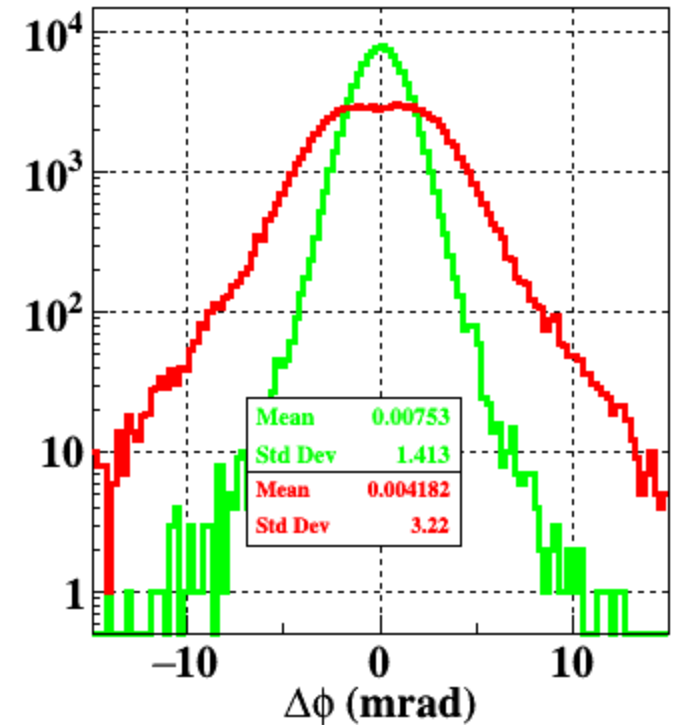
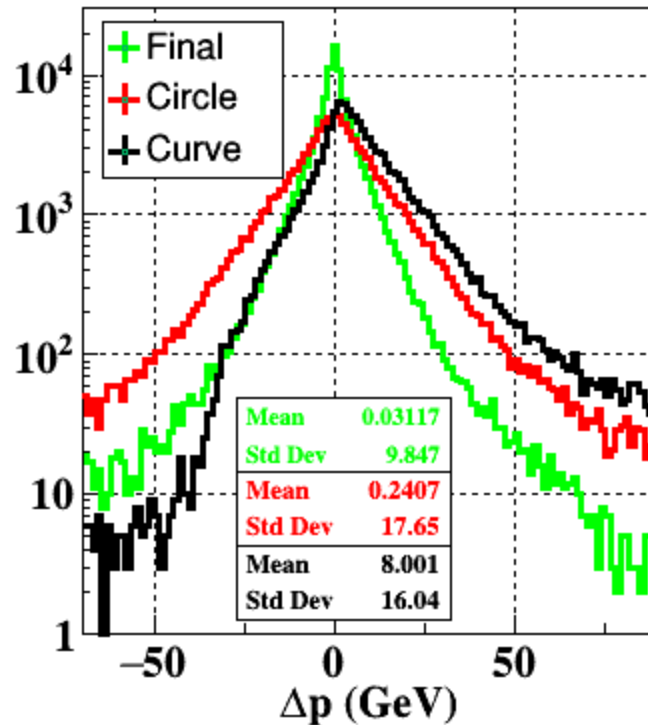
- In a pixel detector or in a double-sided silicon strip detector, m is two dimensional.
- In a one-sided silicon strip detector, it is one-dimensional.
- In a drift chamber or MWPC with several layers, the measurement may be the result of an internal track reconstruction. In this case the vector m may be four or five-dimensional.



- Strictly, the estimate of track parameters is optimal only if the following assumptions hold:
 - The model is linear.
 - The noise is Gaussian.
 - The covariance matrix of the noise does not depend on the parameters.
- If the model is linear, it is still the best linear estimate.
- In practice, none of the assumptions hold exactly.

Simple example

- Uniform magnetic field along Z-axis, bending is only in X-Y plane,
- Use again the same $\rho = \text{slope} (\cot\theta) \times z + \text{intersect eqn}$, to obtain $\cot\theta$ and z of track. Similarly also able to get ϕ and p_T in by fitting in $r-\phi$ plane.
- Combining these two independent sets obtain full track parameters ($\phi_0, z_0, \cot(\theta), \phi, q/p$) and its error.
- Approximate radius of curvature can be determined using first and last in silicon detector, as well as the centre of the circle,
- Use that and fit eqn of circle in X-Y plane to track parameters in bending plane



Kalman filter and fit technique

A329, 493 CPC96, 189 A241, 115 NIM 176, 29

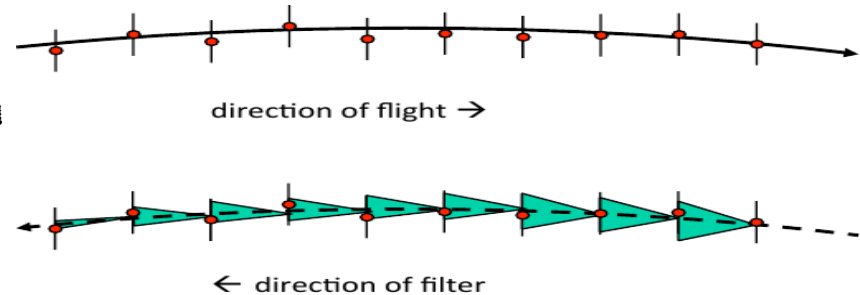
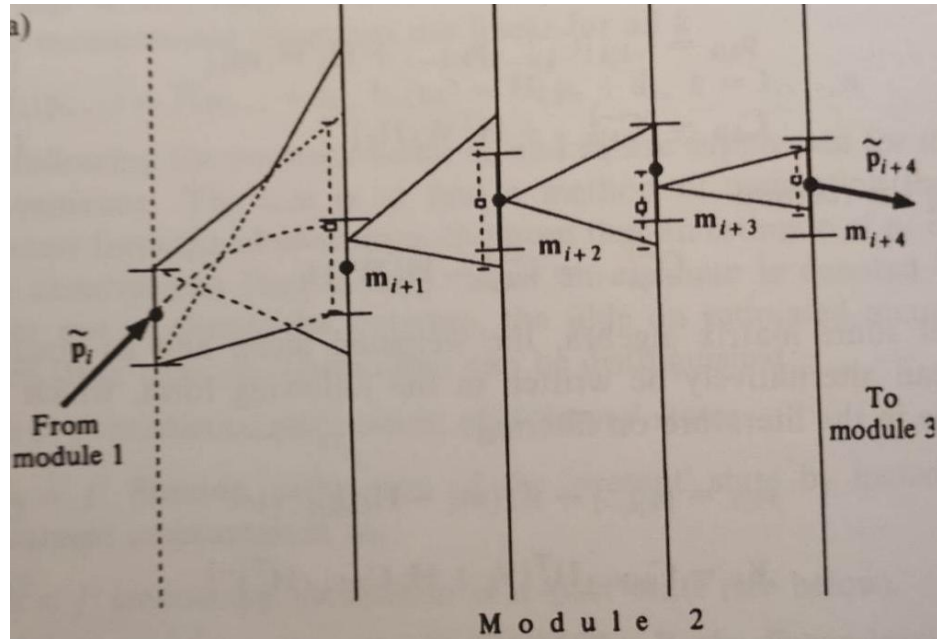
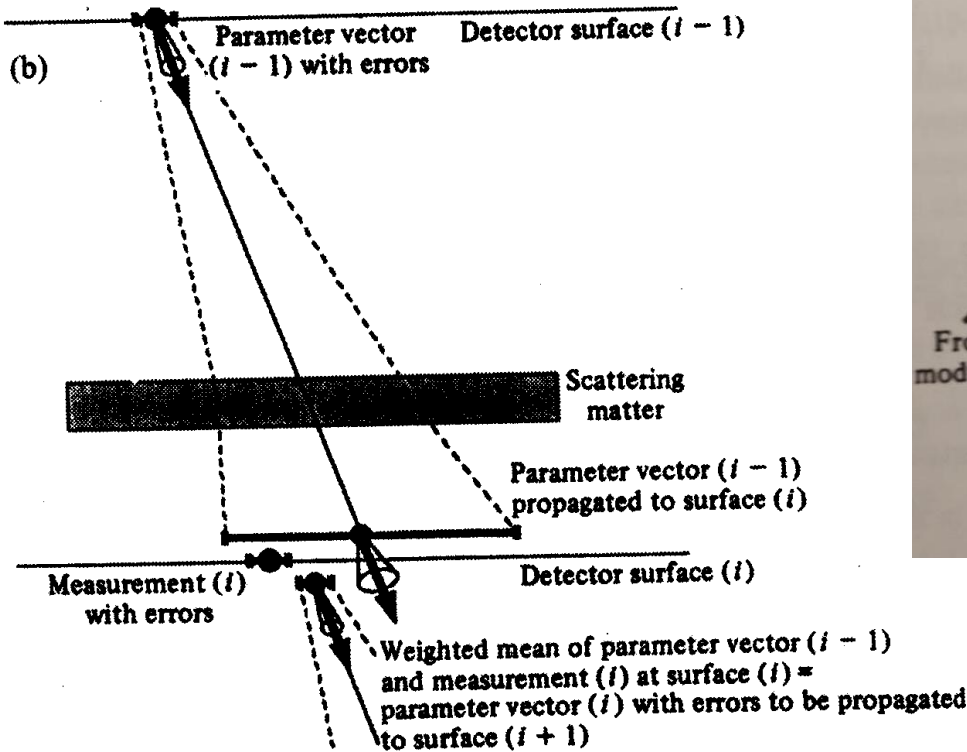
$$\varepsilon = \varepsilon_{\text{detector}} + \varepsilon_{\text{ms}},$$

ε_{ms} is not diagonal. LSM requires inversion of the $n \times n$ covariance matrices and computing time necessary to invert grows as n^3 . The reality is even worse, in presence of ambiguities and outlier

- A recursive or stepwise procedure for estimating the state vectors of a linear or discrete dynamic system (e.g., a track points in a different layer).
- Originally introduced in 1960, and was used in optimal signal processing, navigation, spacecraft tracking
- 1983, P. Billoir first proposed this technique in HEP (did not know that he was deriving Kalman technique)
- Since the early 90's, most of the HEP experiments have moved to this technique.
- Kalman filter brings additional benefits to tracking:
 - Local treatment of multiple scattering
 - Use in local pattern recognition
 - Integrating (non-Gaussian) energy loss in the track model
- Kalman filter also exists for vertex reconstruct

Kalman filter

- Track parameters : $p_k = h_k(p_{k-1}) + P_k \delta_k$, $\langle \delta_k \rangle = 0$, $C(\delta_k) = Q_k$
- Measurement equations, $m_k = f_k(p_k) + \epsilon_k$, $\langle \epsilon_k \rangle = 0$, $C(\epsilon_k) = V_k = W_k^{-1}$
- f_k , maps the track parameters on the measurement surface, is frequently linear, otherwise approximate by its 1st order expansion.
- h_k , in nonlinear in most cases, but Kalman filter in its basic forms assumes a linear system.



What is the track parameters in the first layer ?

Kalman filter

- The state vector at any step is the combination of extrapolation from previous measurement and measurement at the point,

$$\mathbf{p}_k^k = K_k^1 \mathbf{p}_k^{k-1} + K_k^2 \mathbf{m}_k, \quad \mathbf{p}_k^{k-1} = F_{k-1} \mathbf{p}_{k-1}$$

Where K_k^1 and K_k^2 are two weight factors, \mathbf{p}_k^{k-1} is the expected state vector from previous measurements

- Weight factors is calculated (for true state vector, \mathbf{p}) from the minimization of

$$\chi^2 = (\mathbf{m}_k - f(\mathbf{p}))^T V^{-1} (\mathbf{m}_k - f(\mathbf{p})) + (\mathbf{p} - \mathbf{p}_k^{k-1})^T (\mathbf{C}_k^{k-1})^{-1} (\mathbf{p} - \mathbf{p}_k^{k-1})$$

- Which has the following matrix algebra in each step

$$V = (V_k + A_k \mathbf{C}_k^{k-1} A_k^T), \quad A_k = \partial f / \partial \mathbf{p}_k$$

$$\mathbf{C}_k^{k-1} = F_{k-1} \mathbf{C}_{k-1} F_{k-1}^T + \mathbf{Q}_{k-1}$$

$$K_k = \mathbf{C}_k^{k-1} A_k^T (V_k + A_k \mathbf{C}_k^{k-1} A_k^T)^{-1}$$

$$\mathbf{p}_k = F_{k-1} \mathbf{p}_{k-1} + K_k (\mathbf{m}_k - A_k F_{k-1} \mathbf{p}_{k-1}) = (I - K_k A_k) \mathbf{p}_k^{k-1} + K_k \mathbf{m}_k$$

$$\mathbf{C}_k = (I - K_k A_k) \mathbf{C}_k^{k-1}$$

Derive these expressions from the minimization of χ^2

Kalman filter

- Which has the following matrix algebra in each step

$$V = (V_k + A_k C_k^{k-1} A_k^T)$$

$$C_k^{k-1} = F_{k-1} C_{k-1} F_{k-1}^T + Q_{k-1}$$

$$K_k = C_k^{k-1} A_k^T (V_k + A_k C_k^{k-1} A_k^T)^{-1}$$

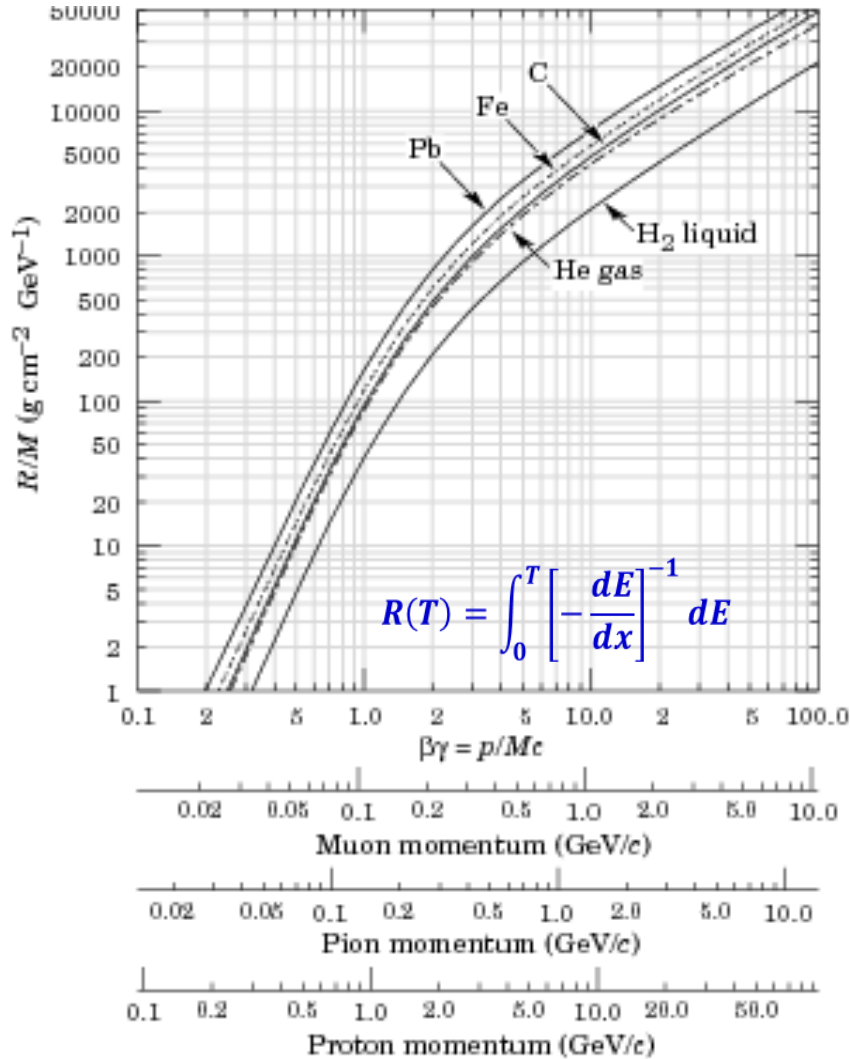
$$p_k = F_{k-1} p_{k-1} + K_k (m_k - A_k F_{k-1} p_{k-1}) = (I - K_k A_k) p_k^{k-1} + K_k m_k$$

$$C_k = (I - K_k A_k) C_k^{k-1}$$

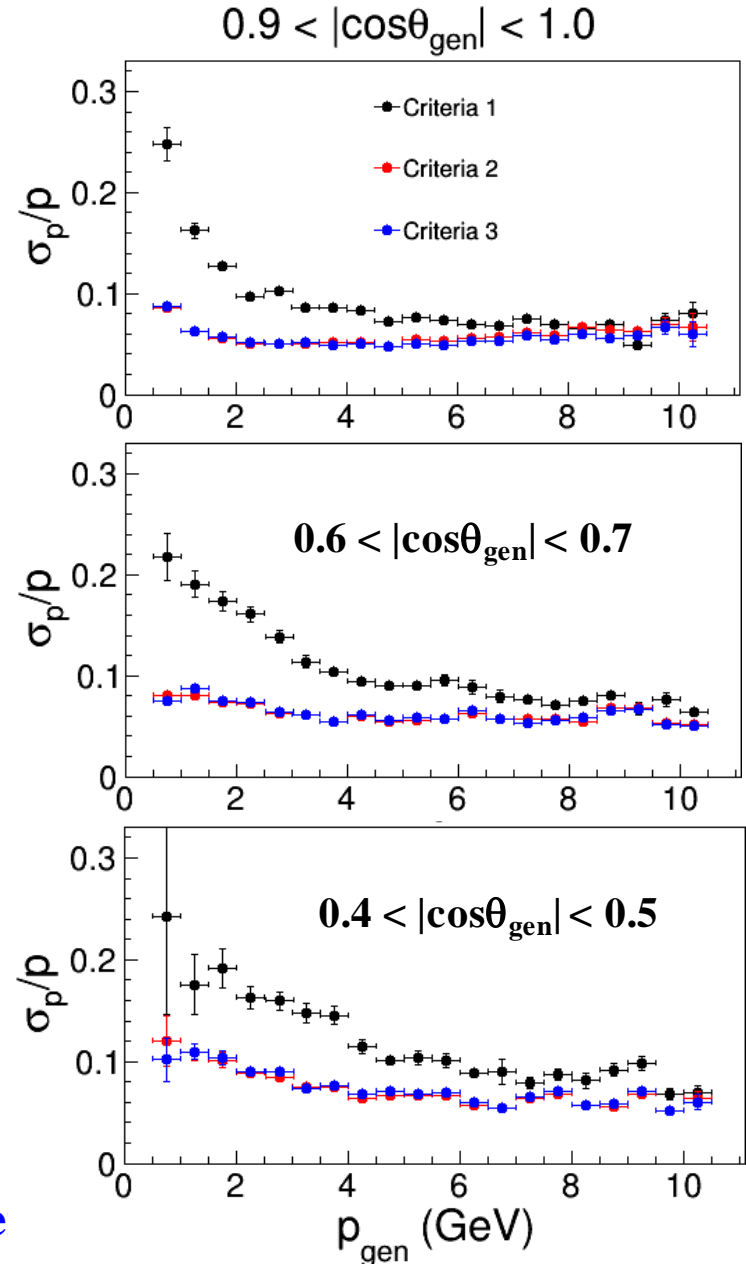
- Where

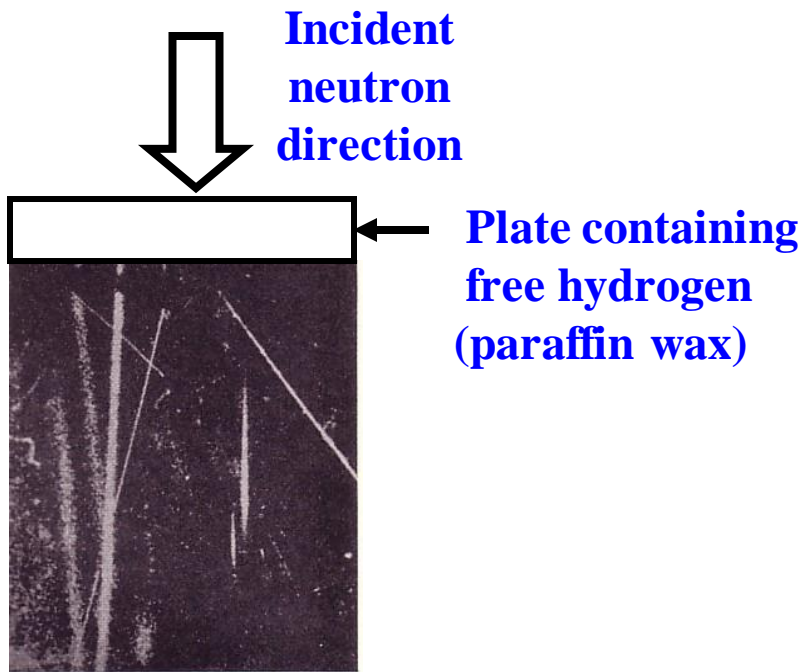
- p_k : State vector $\{\phi_0, dz, \cot(\lambda), \phi, q/p_T\}$, (5×1)
- C_k : Stage covariance matrix, (5×5)
- F_k : Propagator matrix of state vector p_k , (5×5)
- Q_k : Noise matrix due to multiple scattering/ionisation loss, (5×5)
- Some basics are in NIMA 329 (1993) 493
- m_k : (z, ϕ) position measurement, (2×1)
- V_k : Error matrix of m_k , $[z, \phi]$ (2×2)
- A_k : Measurement function ($\partial f / \partial p_k$ from expression $m_k = f(p_k)$), (2×5)
- K_k : Kalman gain factor, (5×2)
- $K_k^1 : (I - K_k A_k)$ and $K_k^2 = K_k$

Momentum without magnetic field



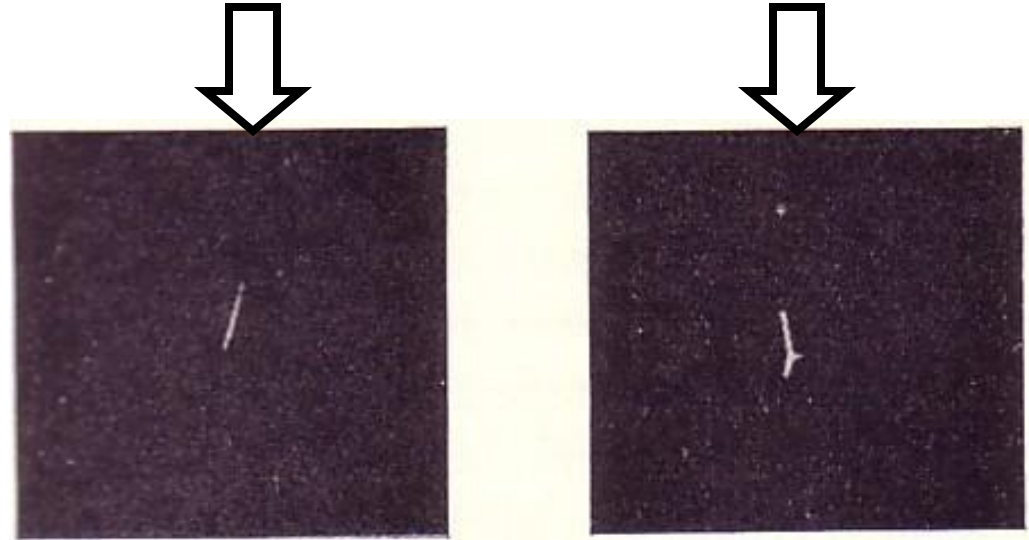
- Path length** : Where measurement of curvature is poor due to large multiple scattering (JINST 13 (2018) no.09, P09015) or poor position measurement





proton tracks ejected from paraffin wax

Neutron mass



Recoiling Nitrogen nuclei

Assume that incident neutral radiation consists of particles of mass m moving with velocities $v < V_{\text{mx}}$

Determine maximum velocity of recoil protons (U_p) and Nitrogen nuclei (U_N) from maximum observed range

$$U_p = \frac{2m}{m + m_p} V_{\text{mx}}$$

$$U_N = \frac{2m}{m + m_N} V_{\text{mx}}$$

From non-relativistic energy-momentum Conservation m_p : proton mass; m_N : Nitrogen nucleus mass

➔
$$\frac{U_p}{U_N} = \frac{m + m_N}{m + m_p}$$

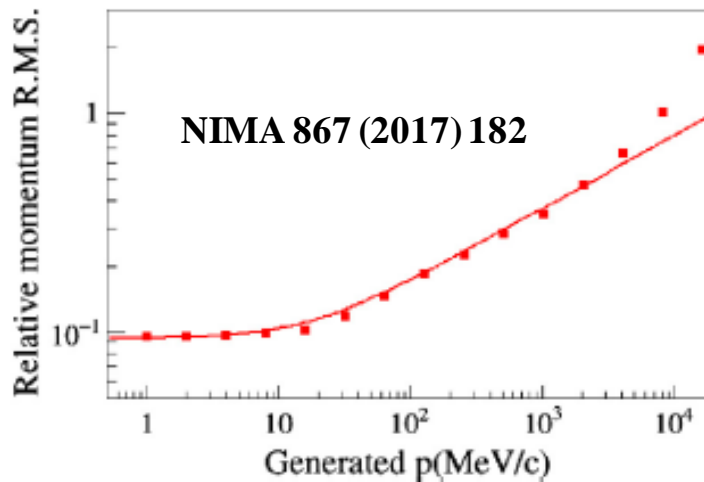
From measured ratio U_p / U_N and known values of m_p , m_N determine neutron mass: $m \equiv m_n \approx m_p$

Present mass values : $m_p = 938.272 \text{ MeV}/c^2$; $m_n = 939.565 \text{ MeV}/c^2$

Momentum without magnetic field

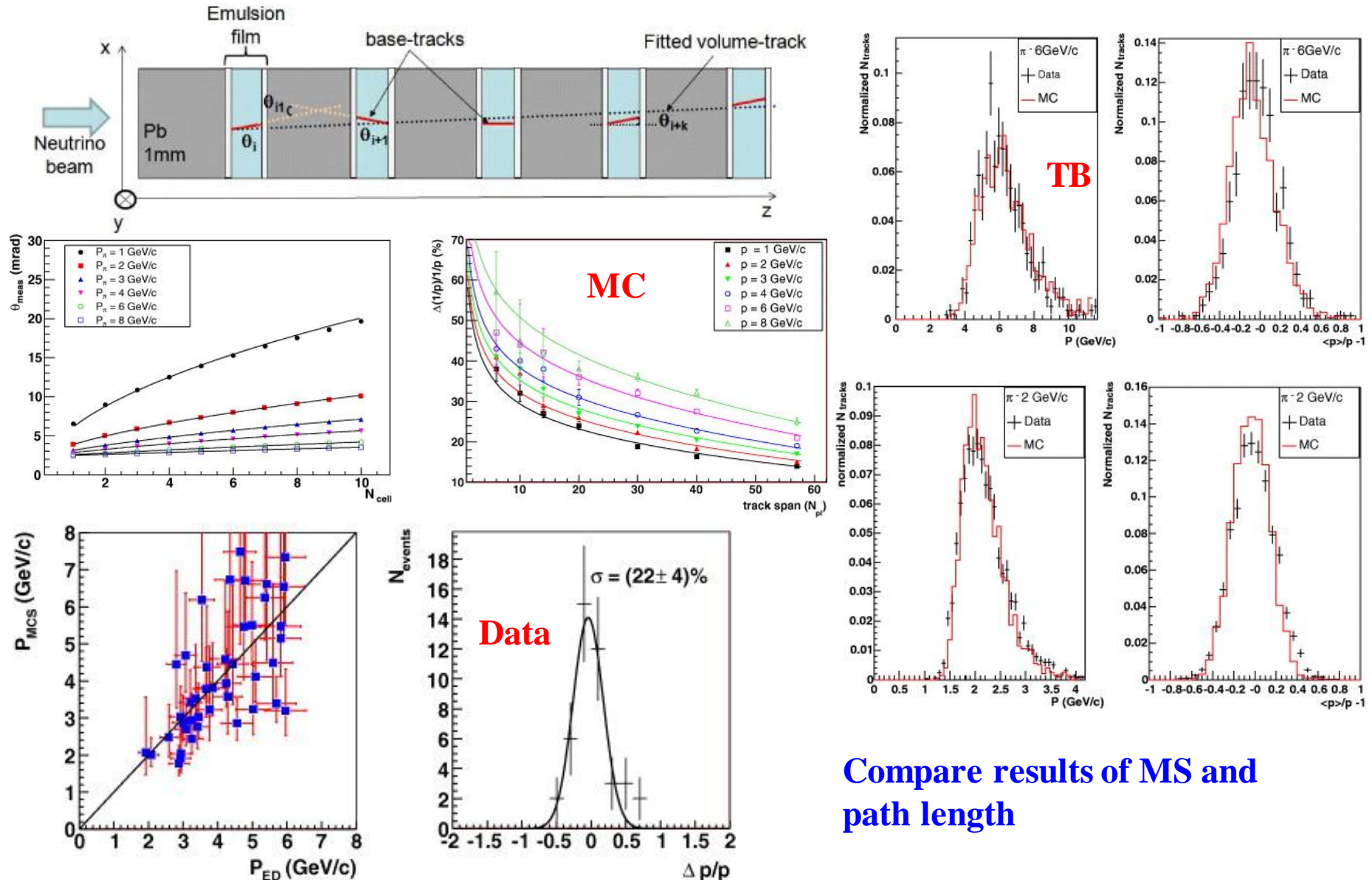
- Use angle of multiple scattering (NIMA 867 (2017) 182, concept for LAr TPC)
- Bayesian analysis on a series of Kalman filters

$$\theta_0 \approx \frac{13.6 \text{ MeV} / c}{\beta c p} \sqrt{\frac{x}{X_0}} \left[1 + 0.38 \ln \left(\frac{x}{X_0} \right) \right]$$



Momentum using angle of multiple scattering

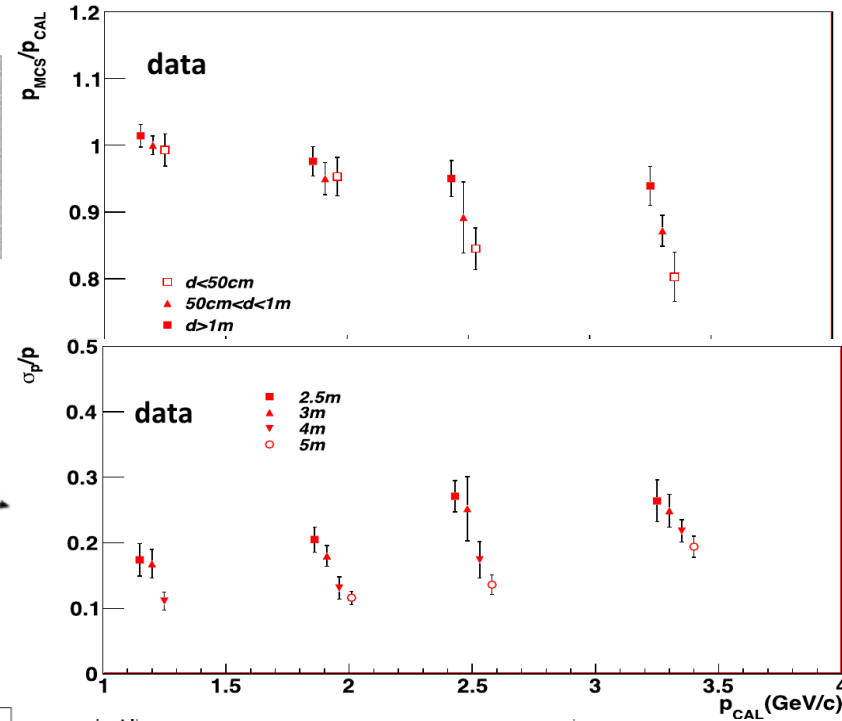
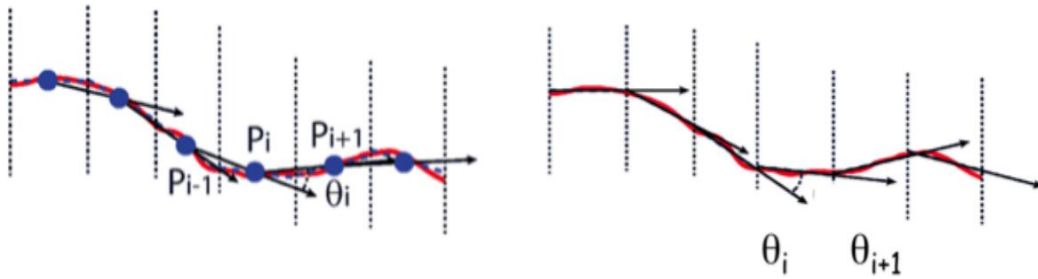
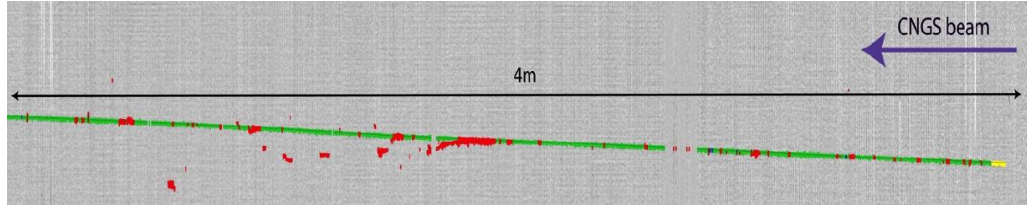
- New J Phys 14 (2012) 013026 (OPERA Lead-emulsion target)



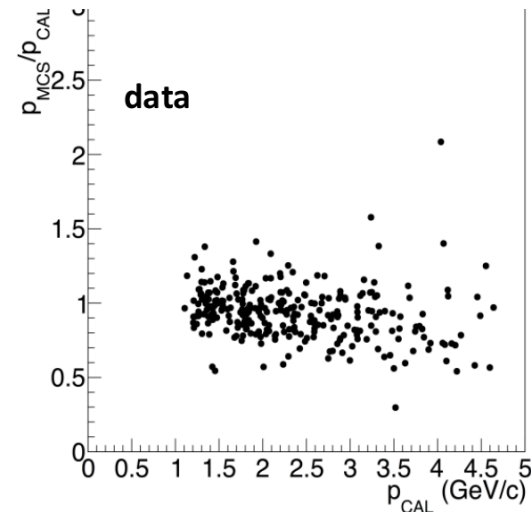
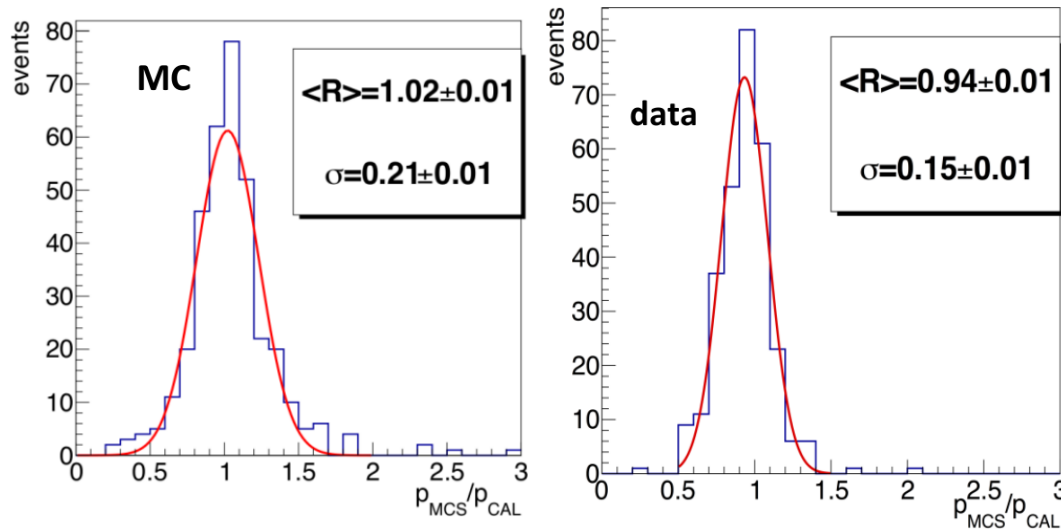
Compare results of MS and path length

Momentum using angle of multiple scattering

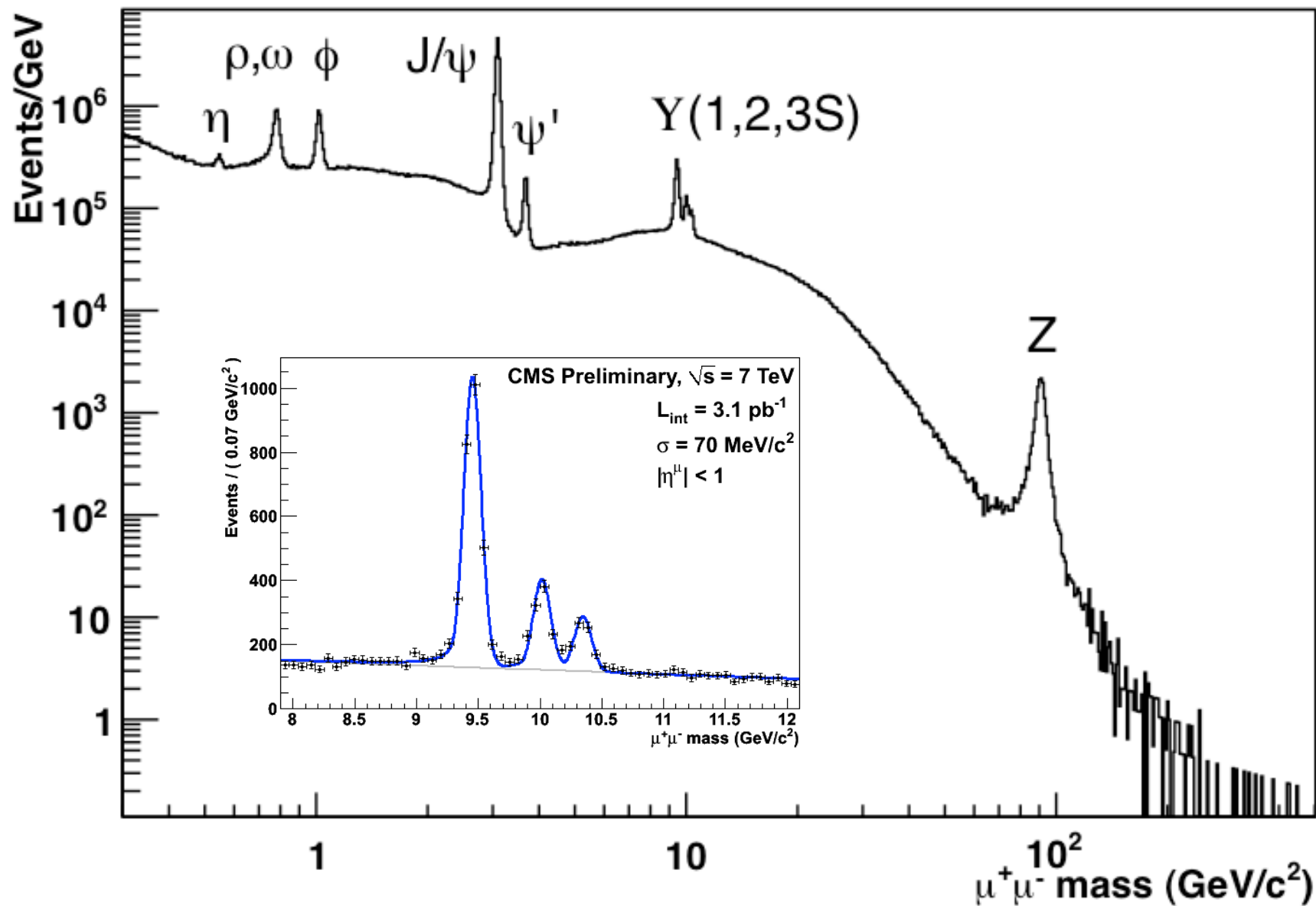
- JINST 12 (2017) no.04, P04010 (ICARUS ~1kt T600 LAr TPC)



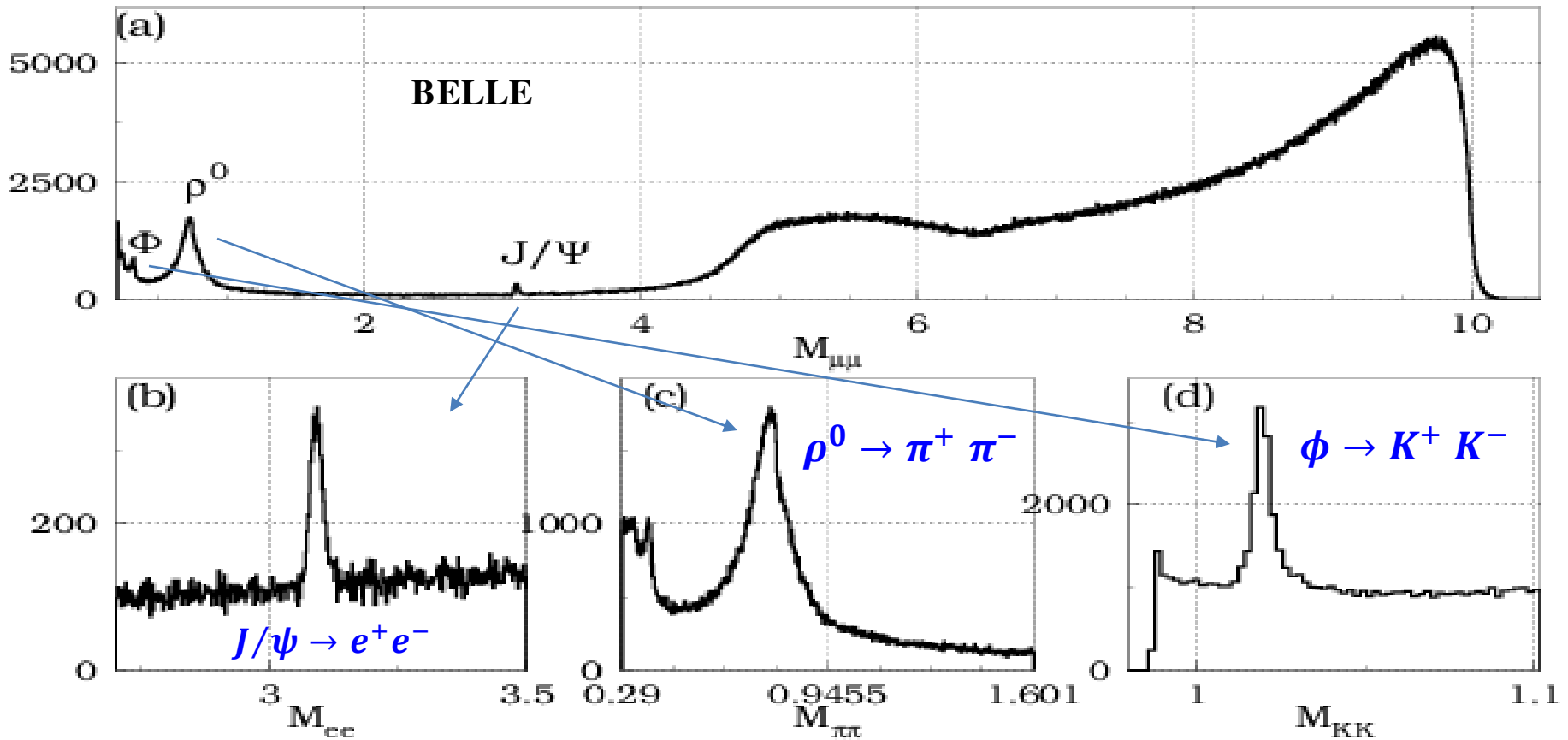
Compare results of MS and path length



Calibration of tracking system over a wide ranges of momentum

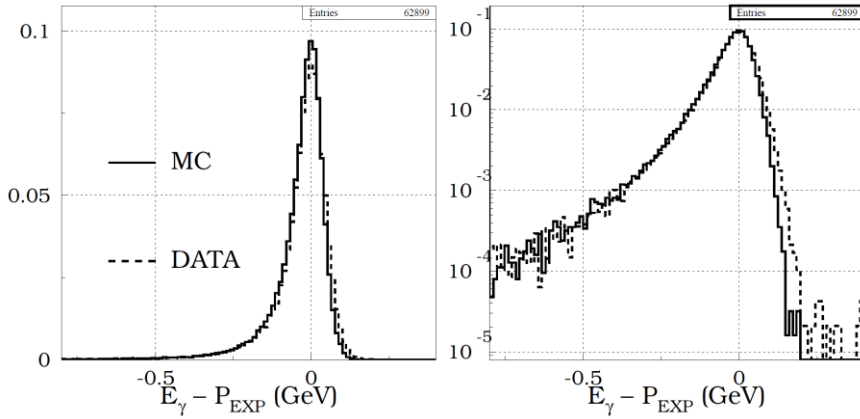
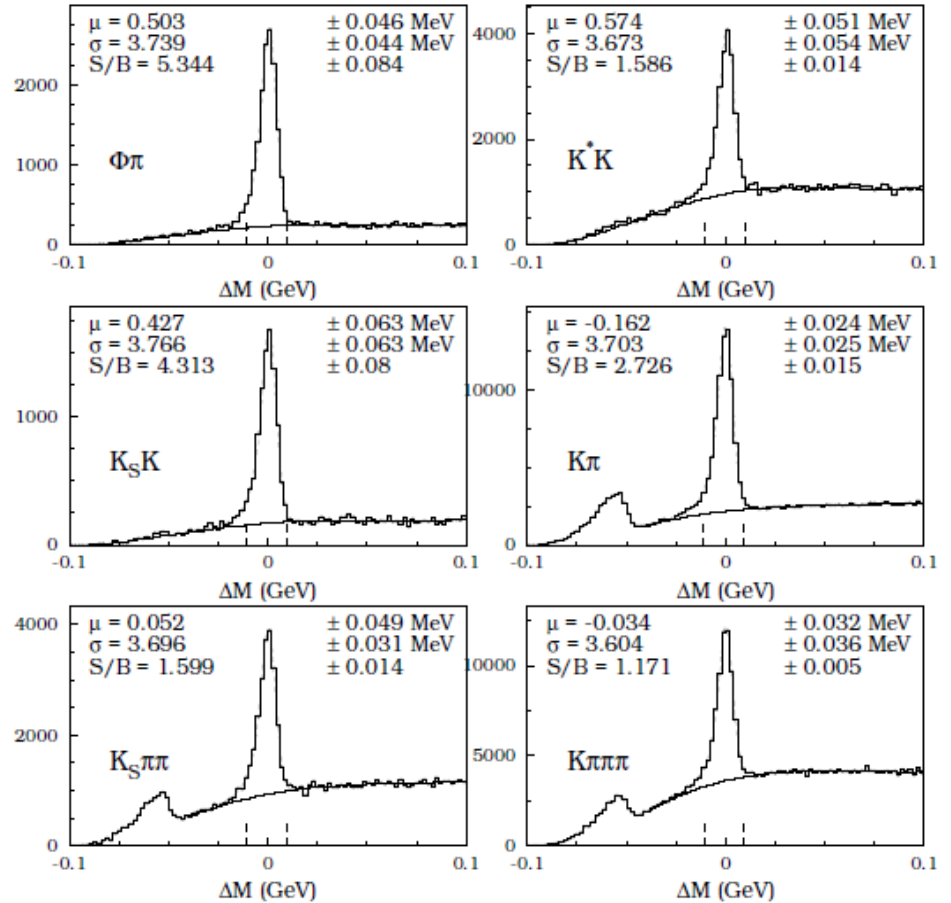
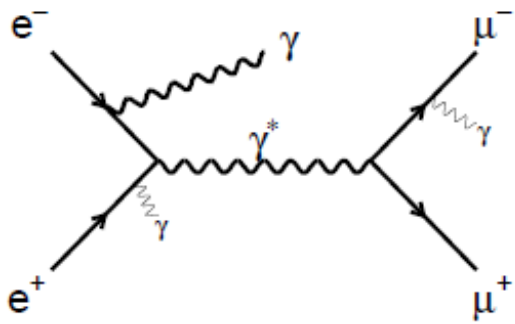


Calibration through mass peak

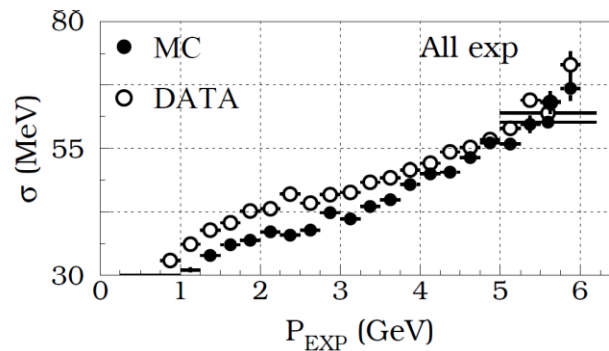
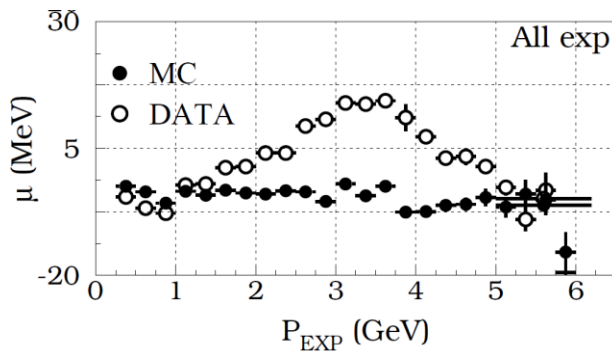


- Similarly, calibration of proton using, $\Lambda (\rightarrow p\pi^-)$ mass peak
- Search for peak in $Z \rightarrow \mu^+\mu^-\gamma$ for calibration of γ (ECAL)
- $\gamma + jet$: For the calibration of hadronic jets

ECAL calibration with data (BELLE)



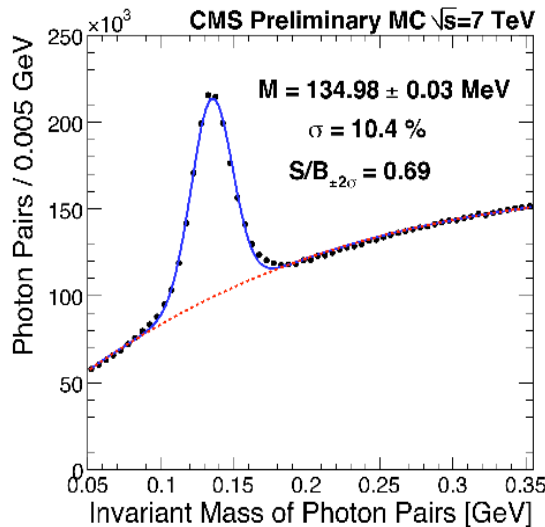
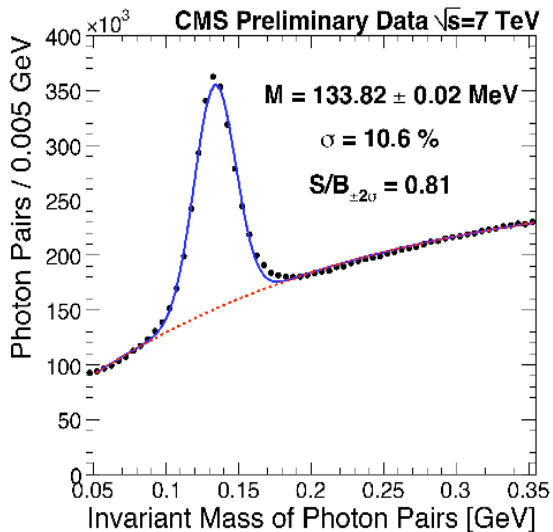
- Data was corrected using these two calibrations along with $\pi^0 \rightarrow \gamma\gamma$



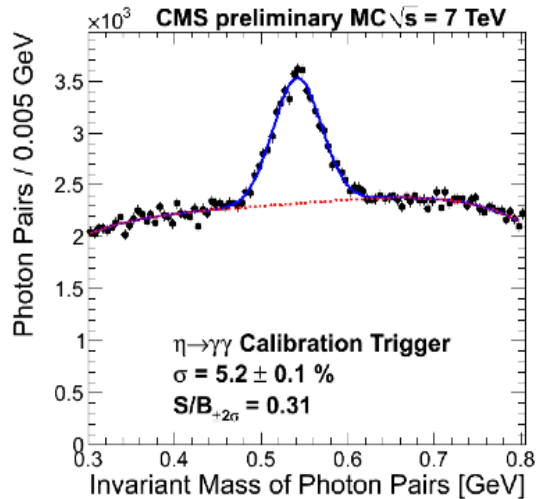
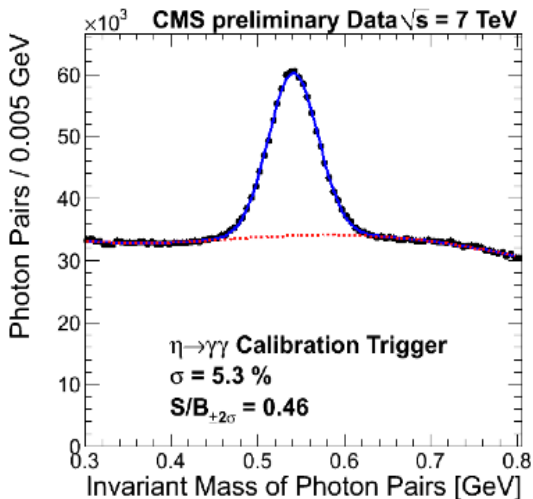
for $D^0 \rightarrow k^-\pi^+\pi^0$, $D^0 \rightarrow k^-\pi^+\pi^+\pi^-$, $D^0 \rightarrow k^-\pi^+\pi^-\pi^+$, $D^0 \rightarrow K_S^0 K^+\pi^-\pi^+$ for photon energy greater than 300 MeV.
 | data : EXP07-EXP19.

$$D^*(s) \rightarrow D(s)\gamma$$

π^0 / η calibration



“Off-line selection”: Kinematics: $P_T(\gamma) > 0.4$ GeV, $P_T(\gamma\text{pair}) > 1$ GeV.

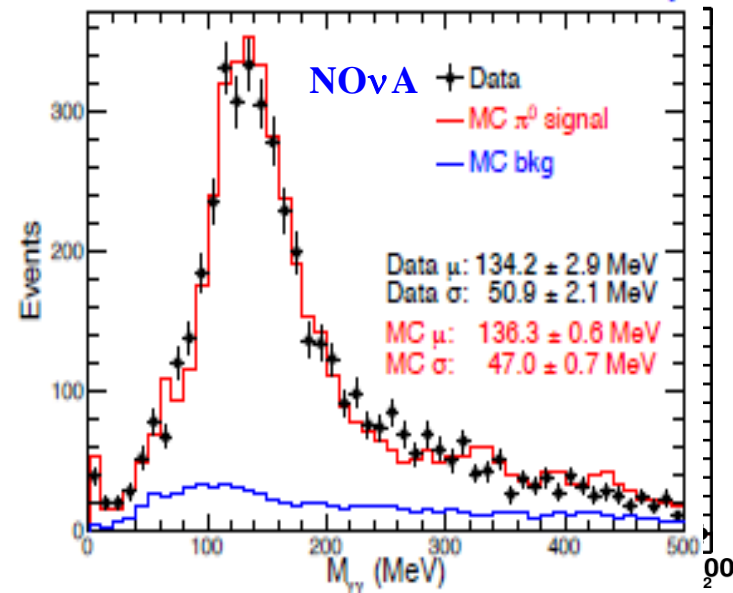


Kinematics: $P_T(\gamma) > 0.8$ GeV, $P_T(\gamma\text{pair}) > 3$ GeV

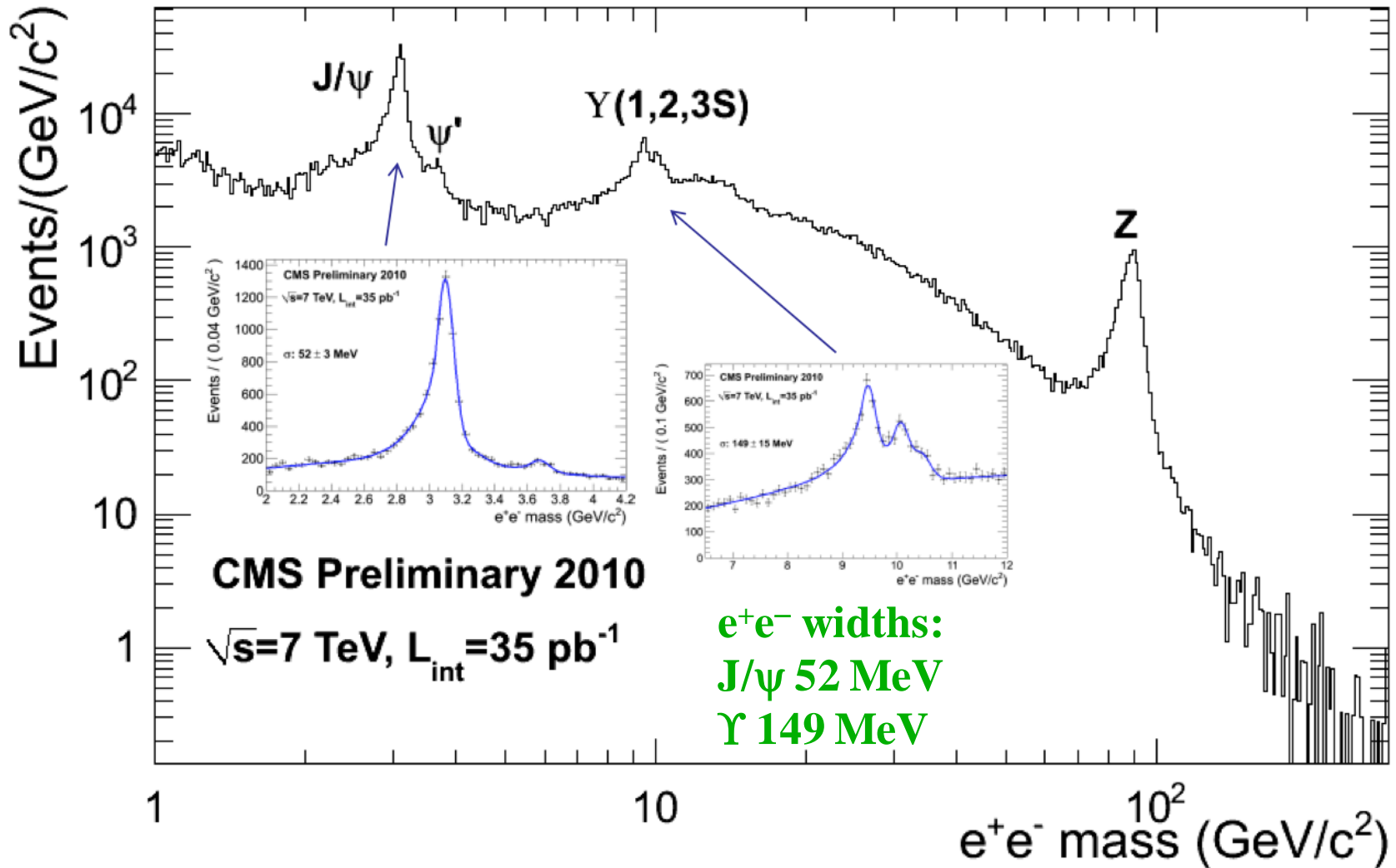
• Pros :

- High Statics,
 - Crystal by crystal intercalibration,
 - Energy scale calibration
- ## • Cons :
- Reco of low energy γ only,
 - High energy γ overlap,
 - Sizable background

• Calibrated photon energy
 $\Rightarrow \pi^0 / \eta$ mass peak at right position and minimum spread



ECAL Calibration: High Energy



Hands-on Session

- **Using input from the track finder algorithm and other information**
 - **Initialise the track parameters**
 - **Estimate the track parameters by imposing least square fit**
 - **Compare with generator level information**

- **For future**
 - **Using simple equation of circle, calculate the transverse momentum of muon tracks**
 - **Include the ionization energy loss and multiple scattering in the fit to improve resolutions**
 - **Add more noise and inefficiency and optimized the track finder algorithm**
 - **Kalman filter**
 - **.....**