Unbiased Exponential Resummation : New way of Exploring finite-density QCD

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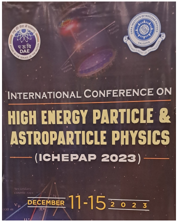
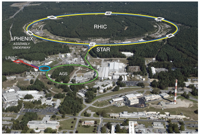
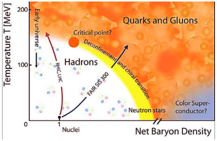


Why finite-density QCD?





- Dynamics in colliders like **RHIC** ≡ finite-density, where wide range of heavy ions collide, producing **high density** of baryons at **high temperatures**
- Understand **QCD phase diagram**, quarks, gluons in $\underline{\mu_B, n_B \neq 0}$ regime
- (**Crossover** $T =$ **156.5 (1.5)** MeV for $\underline{\mu_B = 0}$ from **Lattice QCD**) [HotQCD, Bazavov et. al., Phys. Lett. B 795, 15 (2019)]
- Still **mostly conjectured**. Need evidences of : 1. **Nature** of phase transition between **QGP** (**high** T, μ_B) and present **hadronic** phase (**low** T, μ_B) and its **order** (scaling analysis), 2. QCD **critical point** (if any) 3. $\underline{T_{chiral}, T_{conf}}$ and their differences, 4. **neutron stars**, 5. **color superconductors** etc.
- For all these, know QCD **equation of state** and thermodynamics, **partition function** $\boldsymbol{\mathcal{Z}(\mu, V, T)}$, in a **non-perturbative** formulation.

The real trouble

- Lattice QCD Intro - Talk by **Marina Marinkovic** on 11.12.2023
- Grassmann integration $\rightarrow \mathcal{Z}(\mu_B) = \int \mathcal{D}U\, e^{-S_g[U]}\, [\det \mathcal{M}(\mu_B, U)]$
- Except $\mu_B = 0$, determinant at **finite real** μ_B is **complex** $\rightarrow$ Monte-Carlo sampling (MCS) **not possible** for complex integral measure.
- For applying MCS, we **re-weight** the measure : Make **measure real**, and the **observable complex**. And how we do it ??
- $\mathcal{Z}(\mu_B) = \int \mathcal{D}U e^{-S_g[U]} [\det \mathcal{M}(0, U)] \mathcal{A}(\mu_B, U)$ where observable $\mathcal{A}(\mu_B, U) = \det \mathcal{M}(\mu_B, U) / \det \mathcal{M}(0, U)$ and is complex
- $\mathcal{A}(\mu_B) = R(\mu_B)\, e^{i\boldsymbol{\theta}(\boldsymbol{\mu_B})}$ and $\theta \rightarrow$ **sign problem** : **Large oscillations** between $+$ and $-$ due to large R and θ for large μ_B, computing not easy.
- $\langle \cos\theta \rangle \approx 0$: **average phasefactor** $\langle \cos\theta \rangle = \frac{1}{N_{conf}} \sum_U \cos\theta(U)$
- Many methods proposed to avoid this problem (imaginary μ analysis, Lefschitz thimbles, contour deformation), here I focus on **Approach of Taylor expansion around $\mu_B = 0$ and subsequent resummation of the series**

Taylor Expansion : good but that's not all

- Two thermodynamic observables : 1) $\boldsymbol{\frac{\Delta P}{T^4}}$ and 2) $\boldsymbol{\frac{\mathcal{N}}{T^3}} = \frac{\partial}{\partial(\mu_B/T)}\left[\frac{\Delta P}{T^4}\right]$
- Taylor Expansion to $\mathcal{O}(\mu_B^{\boldsymbol{N}})$ of ΔP given by **[Ejiri** et.al. hep-lat/**0312006]**

$$\frac{\Delta P_N^{\mathrm{T}}}{T^4} = \frac{1}{VT^3}\ln\left[\frac{\mathcal{Z}(\mu_B)}{\mathcal{Z}(0)}\right] = \sum_{n=1}^{N} c_n \left(\frac{\mu_B}{T}\right)^n \tag{1}$$

- **Particle-antiparticle symmetry** of QCD: Eqn.(1) is **even** in μ_B/T.
- $c_n \sim \sum \langle (D_a)^p (D_b)^q \cdots \rangle$: **Linear combinations**
- D_n are **correlation functions** $\equiv \frac{\partial^n}{\partial(\mu_B/T)^n} \ln\det\mathcal{M}(\mu_B)\big|_{\mu_B=0}$,

$$c_1 = \langle D_1 \rangle\,, \quad c_2 = \langle D_2 \rangle + \left\langle D_1^2 \right\rangle \quad \cdots$$

- CP symmetry: D_n are purely **real** (**imaginary**) for n being **even** (**odd**)
- All powers D_a^p, D_b^q and products $D_a^p\, D_b^q$ are **unbiased** (discussed later)

Drawbacks of Taylor and so ...

Good but,

1. **Finite** order in μ_B
2. **Slowly converging non-monotonic** behaviour for different N
3. Requires computing **higher** orders, which becomes extremely **tedious**.

Resummation of finite-order series is a good alternative because,

- A series to **all orders** in μ_B and get **all** the contributions of D_n $\underline{n \leq N}$.
- Results :- **Improved** monotonicity and convergence (needs **more work**)
- Some knowledge about **higher** c_n from **lower ones**, which is good (**more precision** in c_2 and c_4)

And these are achieved by using

1. **Rational functions** in **Paḋe** resummation [Bollweg et.al., **Phys.Rev.D 105** (2022) **7**, **074511**] and
2. **Exponential function** in **exponential** resummation (**I will discuss**)

Exponential Resummation and Stochastic Bias

$$\frac{\Delta P_N^{\mathrm{R}}}{T^4} = \frac{1}{VT^3} \ln \left\{ \mathrm{Re} \left[\left\langle \exp \left[\sum_{n=1}^{N} \left(\frac{\mu_B}{T} \right)^n \frac{\bar{D}_n}{n!} \right] \right\rangle_0 \right] \right\} \quad \text{(real part only)} \tag{2}$$

$$\bar{D}_n = \frac{1}{N_R} \sum_{r=1}^{N_R} D_n^{(r)}, \quad \text{where} \quad D_n^{(r)} \equiv \frac{\partial^n}{\partial(\mu_B/T)^n} \ln \det \mathcal{M}^{(r)}(\mu_B) \Big|_{\mu_B=0} \tag{3}$$

- **Analytically not possible** to determine $\mathcal{M}^{-1}$ $\left[D_n = \sum \mathrm{tr}\left(f_n\left(\mathcal{M}^{-1}\right)\right)\right]$. So, use **random volume sources**. We use N_R number of random sources within **every** gauge configuration with $D_n^{(r)} \rightarrow$ estimate of D_n from source r.

1. **Finite** $N_R \rightarrow$ **estimate bias** in each D_n, **vanishes** as $N_R \rightarrow \infty$
2. Eqns.(2) and (3) $\rightarrow$ **biased estimates** of $(D_n)^m$ for $\boldsymbol{m \geq 2}$. (**greater** effect for **larger values, orders** in μ_B)

$$\left(D_n\right)^m = \text{Biased estimate} + [D_n^m] \quad \text{where} \quad [D_n^m] = \sum_{r_1 \neq \ldots \neq r_m}^{N_R} \cdots \sum^{N_R} D_n^{(r_1)} \cdots D_n^{(r_m)}$$

All m **distinct sources** on **equal** footing in **unbiased estimate** $[D_n^m]$

Way out: Cumulant expansion

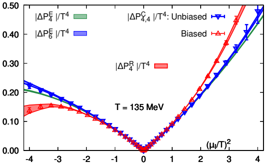
To **eliminate** bias, we first use cumulant expansion. In **cumulant expansion** of ΔP with <u>M cumulants</u>, we find

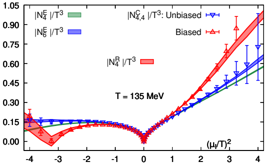
$$\frac{\Delta P^C_{N,M}}{T^4} = \frac{1}{VT^3} \ln \left\langle e^X \right\rangle = \frac{1}{VT^3} \sum_{n=1}^{M} \frac{\kappa_n}{n!}, \qquad X \equiv \sum_{n=1}^{N} \left(\frac{\mu_B}{T}\right)^n \frac{\bar{D}_n}{n!} \tag{4}$$

$\kappa_1 = \langle X \rangle, \kappa_2 = \langle X^2 \rangle - \langle X \rangle^2, \cdots$ ($M, N = 4$ in our work)

Unbiased cumulants : $\kappa_1^u = \kappa_1,\ \kappa_2^u = \kappa_2 + \left\langle [X^2] - X^2 \right\rangle \cdots$

[**SM**, Hegde, Schmidt, **Phys.Rev.D 106** (2023), **034504**, arXiv: **2205.08517**[hep-lat]]





- **Noticeable** difference between unbiased and biased (**blue** vs **red** points)
- But **no all-order resummation** due to **truncation**

Unbiased exponential resummation

Find formulation which will keep **exponential function + unbiased estimates**

$$\frac{\Delta P_N^{\mu_B}}{T^4} = \frac{1}{VT^3}\ \ln\ \mathcal{Z}_N^{\mu_B}, \quad \mathcal{Z}_N^{\mu_B} = \left\langle e^{A(\mu_B)} \right\rangle, \quad A(\mu_B) = \sum_{n=1}^{\boldsymbol{N}} \left(\frac{\mu_B}{T}\right)^n \frac{\mathcal{C}_n}{n!}$$

$$\text{where} \quad C_1 = D_1, \quad C_2 = D_2 + \left(\left[D_1^2\right] - D_1^2\right), \cdots \leftarrow (\boldsymbol{\mu}\ \textbf{basis}) \tag{5}$$

$$\frac{P_N^X}{T^4} = \frac{1}{VT^3}\ \ln\ \mathcal{Z}_N^X, \quad \mathcal{Z}_N^X = \left\langle e^{Y(X)} \right\rangle, \quad Y(X) = \sum_{n=1}^{\boldsymbol{M}} \frac{\mathcal{L}_n(X)}{n!}$$

$$\text{where} \quad X = \sum_{n=1}^{N} \left(\frac{\mu_B}{T}\right)^n D_n, \quad \mathcal{L}_1 = X, \quad \mathcal{L}_2 = \left[X^2\right] - X^2, \cdots \leftarrow (\boldsymbol{X}\ \textbf{basis}) \tag{6}$$

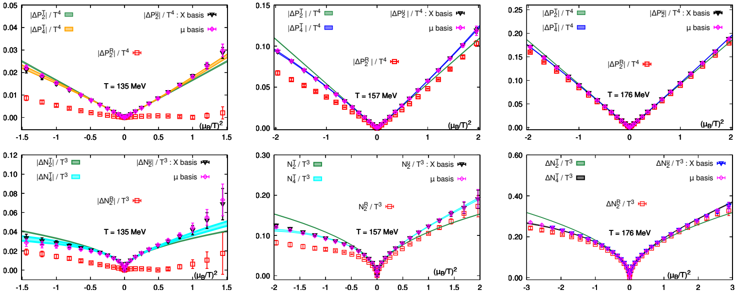
[**SM**, P. Hegde, **Phys.Rev.D 108** (2023) **3**, **034502**, arXiv: **2302.06460**[hep-lat]]

Eqn.(5) → **Taylor series** exactly upto $\mathcal{O}(\mu_B^{\boldsymbol{N}})$

Eqn.(6) → **cumulant expansion** exactly of $\boldsymbol{M}^{th}$ order

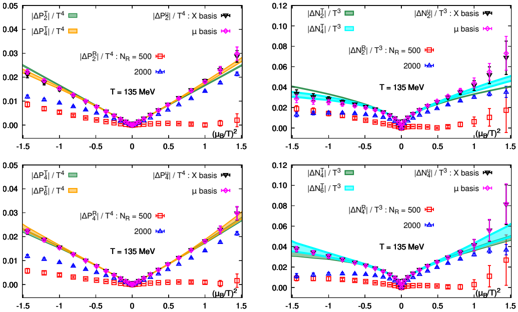
Plots

We do our calculations on a $\mathbf{32^3} \times \mathbf{8}$ lattice with physical quark masses



- **Bias** effect **dominant** in **lower** T (**135** $>$ **157** $>$ **176**)
- **Good** agreement with the **Taylor** results at **all three working** temperatures (Hadronic, **crossover**, **QGP** phases) for **both** ΔP and $\mathcal{N}$
- **Higher** errorbars with higher **values** and **derivative order** of μ_B ($\boldsymbol{\mathcal{N}} > \boldsymbol{\Delta P}$). Also for (**135** $>$ **157** $>$ **176**)

With 2000 random sources

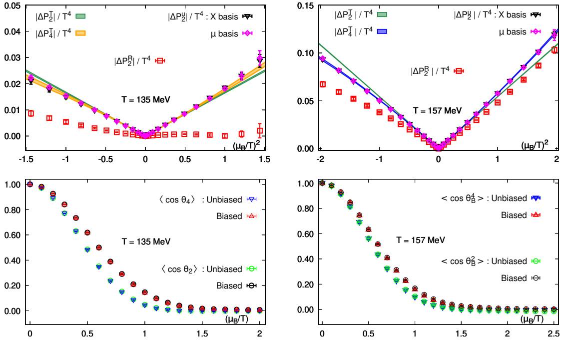


$\Delta P_{2,4}/T^4$ (left) and $\mathcal{N}_{2,4}/T^3$ (right) plots in both bases for $T = 135$ MeV

[**SM**, P. Hegde, arXiv:**2209.11937**[hep-lat]]

- With $\underline{N_R = \mathbf{2000}}$, **old** results improve a lot (**blue** vs **red**)
- **Better** Agreement: $N_R = \mathbf{500}$, **new** method > $N_R = \mathbf{2000}$, **old** method (**black** and **magenta** vs **blue**)
- Saves considerable amt. of **computational time**, **storage space**

Phasefactor plots



- ΔP **not** well-defined beyond $\mu_B/T > \mathbf{1.2}$ for **135**, **1.4** for **157** MeV
- **Exactly** where $\langle \cos\theta \rangle$ starts **converging towards zero** (odd D_n's)
- Falls **faster** in **unbiased** case (**more** oscillations of higher D_n)
- Falls **faster** for **lower** temperatures (stochastic fluctuations are **more**)
- A **reliable** indicator of the severity of sign problem (Needs data of more T)

Conclusions and future Works

- Understood the **importance of unbiased estimates**
- **Difference** between biased and unbiased in **cumulant expansion** results
- Realised the very idea of **unbiased exponential resummation**
- Obtained **unbiased** results to some **finite order** in μ_B $(\mathcal{O}(\mu_B^{\boldsymbol{N}}))$
- Even **better** than **old** results with larger N_R
- Thus saving a **lot of computational costs and time**
- Managed to preserve **unbiased** behaviour **within exponential** structure
- Preserving **phasefactor**, a **good** indicator of **sign problem severity**
- In the limit $N \to \infty$, this is **exactly equal** to the **infinite Taylor series**
- Future works: Exponential vs Paḿde comparative study (going on $\cdots$), studies involving chiral symmetry restoration and breaking etc. etc.

THANK YOU ALL FOR YOUR PATIENCE AND ATTENTION