Muon (μ^-) lifetime Measurement

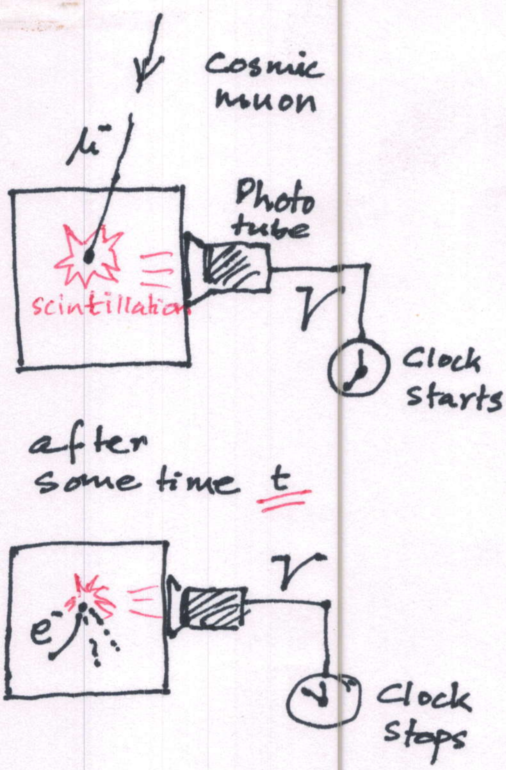
$t = t_{stop} - t_{start}$

is the measured quantity

It is a random quantity

Infinitely many measurements of t is possible.

That infinitely many set is the population



Muon decay lifetime measurement setup.

We know from the theory of muon decay that the decay lifetime (τ) is given by: $\boxed{\tau = \dfrac{192\pi^3\hbar^7}{G_F^2 m_\mu^5 c^4}}$

The measured lifetime, t of one muon is exponentially distributed. i.e.

$$f(t) = \frac{1}{\tau} e^{-t/\tau}$$

The true mean or population mean

$$\int_0^\infty t\, f(t)\, dt = \tau$$ is a constant parameter

This average over infinitely many measurements will give us the true value of muon lifetime and should match exactly with the theory.

This parameter τ has a physical meaning and we are interested in measuring it.

It is impossible to average over infinitely many experiments.

$\Rightarrow$ we can't find the true value of τ.

$\Rightarrow$ Best we can do is estimate τ from a finite number of measurements

e.g. from 10 measurements $t_1, t_2 \cdots t_{10}$ we can estimate τ by taking a finite average

$$\hat{\tau} = \frac{t_1 + t_2 + \cdots + t_{10}}{10}$$

Note

$t_1, t_2 \cdots t_{10}$ are independent and identically distributed. iid

10 is our sample size

$\{t_1, t_2 \ldots t_{10}\}$ is the sample

$\hat{\tau}$, the sample mean is an estimator of τ, the population mean

All possible values of t is the sample space

$\hat{\tau}$ is a function of the sample, the data we have taken $\Rightarrow$ $\hat{\tau}$ is itself a random quantity.

- What is the underlying distribution of $\hat{\tau}$?
- Is the sample mean a "good" estimator of τ?
- How to construct an estimator?
- How to evaluate them?

Theory of point estimation

Definitions, notations ...

$$\vec{x} = \{ x_1, x_2, \dots x_n \}$$

[x can be momentum, energy, decay length ... anything measured]

$\vec{x}$ is a random vector with its own sample space

$\vec{\theta} = \{ \theta_1, \theta_2 \dots \theta_k \}$ is parameter vector.

e.g. if $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$ $\longrightarrow$ $\vec{\theta} = \{ \underset{\theta_1}{\underset{\uparrow}{\mu}}, \underset{\theta_2}{\underset{\uparrow}{\sigma^2}} \}$

$\vec{\theta}$ is constant vector but unknown

$\hat{\vec{\theta}} = \{ \hat{\theta}_1, \hat{\theta}_2 \dots \hat{\theta}_k \}$ are the estimators of $\vec{\theta}$

$\hat{\vec{\theta}} = \hat{\vec{\theta}}(\vec{x})$ are random vectors.

Definition: A point estimator is any function $T(x_1, \dots, x_n)$ of a sample; that is, any statistic is a point estimator

[Casella & Berger]

Distributions of $\vec{x}$ and $\hat{\vec{\theta}}$

$$f_{\text{sample}}(\vec{x}) = f(x_1).f(x_2)\cdots f(x_n) \quad [\text{if } \vec{x} \text{ is iid}$$

$$= \prod_{i=1}^{n} f(x_i)$$

For the muon lifetime exp.

$$f_{\text{sample}}(\vec{t}) = f(t_1)\ldots f(t_n)$$

$$= \frac{1}{\tau^n}\cdot e^{-t_1/\tau} \cdot e^{-t_2/\tau} \cdots e^{-t_n/\tau}$$

$$= \frac{1}{\tau^n}\cdot e^{-\frac{1}{\tau}\left(\sum_{i=1}^{n} t_i\right)}$$

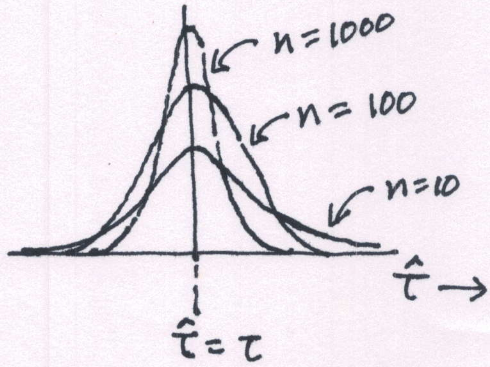
$\hat{\theta}$ is random $\Rightarrow$ has its own distribution called a sampling distribution

In the muon lifetime experiment, using C.L.T.

$\hat{\tau} \sim n\left(\tau, \frac{\tau^2}{n}\right)$ when $n \gg 1$

ie normal distributed with $\mu = \tau$ (mean)

$\sigma^2 = \frac{\tau^2}{n}$ (variance)



Note that

$E[\hat{\tau}] = \tau$ (the true value)

independent of n.

Only the variance becomes narrower.

Such an estimator is called an unbiased estimator

Unbiased and consistent estimators

Suppose the sampling distribution of $\hat{\theta}$ is $g(\hat{\theta};\theta)$

$$\Rightarrow E[\hat{\theta}] = \int \hat{\theta}\, g(\hat{\theta};\theta)\, d\hat{\theta}$$

$$= \int \cdots \int \hat{\theta}(\vec{x})\, f(x_1;\theta) \cdots f(x_n;\theta)\, dx_1 \cdots dx_n$$

1) bias $b = E[\hat{\theta}] - \theta$

does not depend on the measured values of the sample, but sample size.

Unbiased estimator $\Rightarrow b = 0$ independent of n

bias may tend to zero as $n \to \infty$, such an estimator is asymptotically unbiased.

2) As sample size n increases, if the probability of the estimator value $\hat{\theta}$ being different from the true value, tends to zero, i.e.

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \epsilon) = 0 \quad \text{for any } \epsilon > 0$$

We will call the estimator consistent

Consistent $\neq$ unbiased.

3) Mean squared error (MSE)

$$\mathrm{MSE} = E\left[(\hat{\theta}-\theta)^2\right] = E\left[(\hat{\theta} - E[\hat{\theta}] + E[\hat{\theta}] - \theta)^2\right]$$

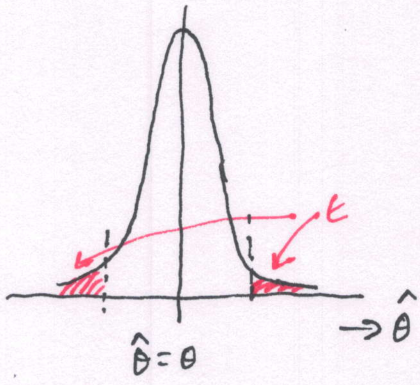
$$\cancel{= E\left[(\hat{\theta} - E[\hat{\theta}])^2\right] +}$$

$$= E\left[(\hat{\theta} - E[\hat{\theta}])^2 + (E[\hat{\theta}] - \theta)^2 + 2(\hat{\theta} - E[\hat{\theta}])(E[\hat{\theta}] - \theta)\right]$$

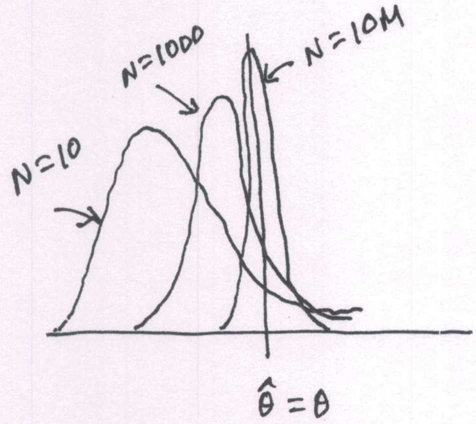
$$= E\left[(\hat{\theta} - E[\hat{\theta}])^2\right] + E\left[(\underbrace{E(\hat{\theta}) - \theta}_{b})^2\right]$$

$$= V(\hat{\theta}) + b^2$$

Can be used as justification for adding statistical and systematic uncertainty in quadrature.



Convergence in probability



Asymptotically unbiased

Estimators of mean, variance, covariance.

Already seen with μ-decay example that,

$$\hat{\mu} \equiv \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is an unbiased and consistent estimator.
– by central limit theorem

[you can also directly calculate $E[\hat{\mu}]$ and show that $E[\hat{\mu}] = \mu$] Homework

Weak law of large numbers (WLLN)

Let x_1, x_2 .. be iid random variables with $E[x_i] = \mu$ and $\mathrm{var}(x_i) = \sigma^2 < \infty$. Then for every $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(|\bar{x}_n - \mu| < \epsilon\right) = 1$$

that is $\bar{x}_n$ converges in probability

This implies consistency of $\bar{x}_n$ by definition

(Note Will not hold for Cauchy or Breit Wigner)

Estimator of variance:

A natural choice would be

$$s'^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$$ ⟵ biased!

Homework Prove that $E[s'^2] = \frac{n-1}{n}\sigma^2$.

$$\Rightarrow \quad s^2 = \frac{n}{n-1}s'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$$

is the unbiased, consistent estimator of σ^2

Note that: s'^2 is consistent too

Proof of $s'^2 = \frac{n-1}{n}\sigma^2$

$$s'^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 = \frac{1}{n}\sum_i (x_i - \mu + \mu - \bar{x}_n)^2$$

$$E(s'^2) = \frac{1}{n}E\left[\sum_i \left\{ (x_i-\mu)^2 + (\bar{x}_n-\mu)^2 - 2(x_i-\mu)(\bar{x}_n-\mu)\right\}\right]$$

$$= \frac{1}{n}\sum_i E\left[(x_i-\mu)^2\right] + \frac{1}{n}\sum_i E\left[(\bar{x}_n-\mu)^2\right]$$

$$- \frac{2}{n}\sum_i E\left[(x_i-\mu)\left(\frac{\sum_{j=1}^{n} x_j}{n} - \mu\right)\right]$$

note that $E\left[(x_i-\mu)^2\right] = \sigma^2$

$E\left[(\bar{x}_n-\mu)^2\right] = \frac{\sigma^2}{n}$ (variance of sample mean)

$$\Rightarrow E[s'^2] = \frac{1}{n}\cdot n\sigma^2 + \frac{1}{n}\cdot n\cdot\frac{\sigma^2}{n} - \frac{2}{n^2}\cdot(\text{term}3)$$

this n comes from $\sum_{i=1}^{n}$

$$\text{term}3 = \sum_i E\left[(x_i-\mu)\left(\sum_j x_j - n\mu\right)\right]$$

$$= \sum_i E\left[(x_i-\mu)\left\{(x_i-\mu) + \sum_{j\neq i}(x_j-\mu)\right\}\right]$$

$$= \sum_i \underbrace{E\left[(x_i-\mu)^2\right]}_{\sigma^2} + \sum_{j\neq i} \underbrace{E\left[(x_i-\mu)(x_j-\mu)\right]}_{\text{covariance of iid } = 0}$$

$$= n\sigma^2$$

$$\therefore E[s'^2] = \sigma^2 + \frac{\sigma^2}{n} - \frac{2}{n^2}\cdot n\sigma^2 = \sigma^2 - \frac{\sigma^2}{n}$$

$$= \frac{n-1}{n}\sigma^2.$$ ✓

If the true population mean is known then

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \underbrace{\overline{x^2}-\mu^2}_{\text{check!}}$$ is unbiased estimator of σ^2

We already know from CLT that the variance of $\overline{x}_n$ is $\frac{\sigma^2}{n}$. It is easy to check directly also,

$$V[\overline{x}_n] = E[\overline{x}^2] - (E[\overline{x}])^2$$

Substitute for $\overline{x} = \frac{1}{n}\sum_i x_i$ and expand.
(see Cowan)

It is not surprising that the estimator of k^{th} central moment

$$m_k = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x}_n)^k.$$

Variance of m_2 that is s^2 is given by

$$V[s^2] = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\mu_2^2\right)$$

For 2 dimensional normal pdf one can show that

$$E[r] = \rho - \frac{\rho(1-\rho^2)}{2n} + O\left(\frac{1}{n^2}\right)$$

$$V[r] = \frac{1}{n}(1-\rho^2)^2 + O\left(\frac{1}{n^2}\right)$$

where $r = \hat{V}_{xy}/s_x s_y$

$$\hat{V}_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x}_n)(y_i-\overline{y}_n) = \frac{n}{n-1}(\overline{xy}_n - \overline{x}_n\cdot\overline{y}_n)$$

$\frac{1}{n}\sum_i x_i y_i$

is the estimator of covariance of x and y