Electroweak Unification and the Standard Model

Lecture 1

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Natural units:

we shall use relativity + quantum mechanics: convenient to set

$$c = 1 \quad \text{and} \quad \hbar = 1$$

Dimensions:

$$\left.\begin{aligned} [c] &= LT^{-1} \Rightarrow \mathrm{L} = \mathrm{T} \\ E &= \hbar\omega \Rightarrow [E] = [T^{-1}] \\ E &= mc^2 \Rightarrow [E] = M \end{aligned}\right\} \quad \mathrm{L} = \mathrm{T} = \mathrm{M}^{-1} = [E]^{-1}$$

conventional to write everything in units of energy (GeV)

$1\ \mathrm{GeV}^{-1} = 0.1973$ fm (1 femtometre = 10^{-15} m)

$1\ \mathrm{GeV}^{-2} = 0.3894$ mb (1 millibarn = 10^{-27} cm^2)

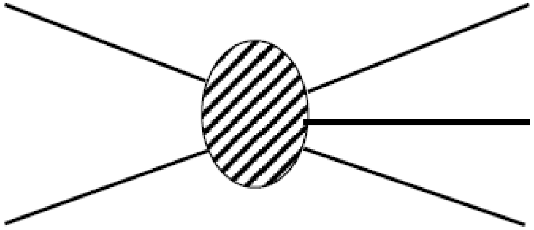
$1\ \mathrm{GeV}^{-1} = 0.6582\mathrm{x}10^{-22}$ s

Thanks to Feynman, we can express the results of RQFT in terms of diagrams which are easy to understand physically. Thus, we can get away, up to a certain point, without learning RQFT.

What can elementary particles do?

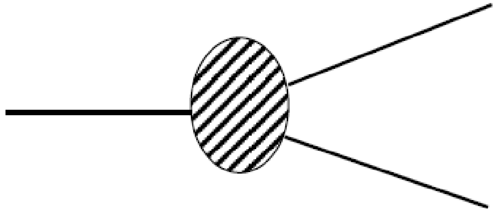
- Scattering processes

  cross-section σ



- Decay processes

  decay width Γ



What makes such processes happen?

Four fundamental interactions:

- gravitation $F \propto G_N m^2$
- electromagnetism $F \propto e^2/4\pi\varepsilon_0$
- strong (nuclear) interaction $F \propto g_s^2$
- weak (nuclear) interaction $F \propto G_F$

At high energies, electromagnetism and weak interactions unify to form the *electroweak interaction*.

Decay processes can be used to determine the nature of the interaction.

A decaying state evolves with time as $e^{-\Gamma t/\hbar}$

i.e. the (mean) lifetime is $\tau = \frac{\hbar}{\Gamma}$

From quantum mechanics, it can be shown that $\Gamma \propto g^2$ where

$$g^2 = \begin{cases} G_N & \text{for gravitation} \\ \dfrac{e^2}{4\pi\varepsilon_0} & \text{for electromagnetism} \\ g_s^2 & \text{for strong interactions} \\ G_F & \text{for weak interactions} \end{cases}$$

Thus, the lifetime of a process satisfies

$$\tau \propto \frac{\hbar}{g^2}$$

i.e. the stronger the interaction, the shorter the lifetime.

Interaction	$\boldsymbol{\tau}$	$\boldsymbol{\ell = c\tau}$
Strong interaction	$\sim 10^{-23}$ s	$\sim 10^{-13}$ cm
Electromagnetic interaction	$\sim 10^{-16}$ s	$\sim 10^{-6}$ cm
Weak interactions	$\sim 10^{-9}$ s	~ 10 cm
Gravitational interaction	$\sim 10^{+22}$ s	$\sim 10^{+34}$ cm

Only weakly-decaying particles will leave observable tracks

Classification of particles according to interactions:

ELEMENTARY PARTICLES

Some well-known particles:

CHARGED LEPTONS:	electron (e^-), muon (μ^-), tau (τ^-)
NEUTRINOS:	electron-neutrino (ν_e), muon neutrino (ν_μ), tau neutrino (ν_τ)
MESONS:	pions (π^+, π^0), kaons (K^+, K^0), rho (ρ^+, ρ^0), eta (η^0), etc.
BARYONS:	proton (p^+), neutron (n^0), Delta ($\Delta^{++}, \Delta^+, \Delta^0$), Lambda ($\Lambda^0$), Sigma ($\Sigma^+, \Sigma^0$), cascade ($\Xi^+, \Xi^0$), Omega-minus ($\Omega^-$)
GAUGE BOSONS:	photon (γ), W-boson (W^+), Z-boson (Z^0), gluons (g)

Each has its own antiparticle:

ANTI-LEPTONS:	positron (e^+), anti-muon (μ^+), anti-tau (τ^+)
NEUTRINOS:	electron-antineutrino ($\bar{\nu}_e$), muon antineutrino ($\bar{\nu}_\mu$), tau antineutrino ($\bar{\nu}_\tau$)
ANTI-MESONS:	pions (π^-, π^0), Kaons ($K^-, \overline{K}^0$), rho (ρ^-, ρ^0), eta (η^0), etc.
ANTI-BARYONS:	antiproton (p^-), antineutron ($\bar{n}^0$), Delta ($\Delta^{--}, \Delta^-, \Delta^0$), Lambda ($\Lambda^0$), Sigma ($\Sigma^-, \Sigma^0$), Cascade ($\Xi^-, \bar{\Xi}^0$), Omega-plus ($\Omega^+$)
GAUGE BOSONS:	photon (γ), W-boson (W^-), Z-boson (Z^0), gluons (g)

Nöther's Theorem and conserved quantum numbers

Consider a system with a Lagrangian $L = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n)$ such that under a transformation $q_i \to q_i + \varepsilon\eta_i$, for all i, the Lagrangian remains unchanged. We call this a symmetry of the system.

It follows that, treating ε as a parameter: $\frac{\partial L}{\partial \varepsilon} = 0$

or, more explicitly,

$$\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i\right) = 0$$

Substituting the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

we get

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\eta_i}{dt}\right] = 0 \qquad \text{or,} \qquad \sum_{i=1}^{n}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\eta_i\right] = 0$$

or,

$$\frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\eta_i\right) = 0$$

i.e. we have a conserved quantity

$$Q = \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\eta_i$$

Thus we have proved Nöther's Theorem: *to every symmetry of a Lagrangian system there corresponds a conserved quantity*. The conserved quantity Q is called the Nöther charge.

Typical conserved quantities and the corresponding symmetries:

Lagrangian Field Theory

Let $\psi(x)$ be a field defined on a Minkowski space with coordinates x i.e. for every value of x there is a value of $\psi(x)$.

If we treat $\psi(x)$ at every point x as a generalised coordinate, then clearly this is a system with *infinite* number of degrees of freedom.

In Lagrangian dynamics, this will be described by a Lagrangian L

$$L = \int d^3\vec{x}\ \ \mathcal{L}\big(\psi(x), \partial_\mu \psi(x)\big)$$

where $\mathcal{L}$ is the Lagrangian density and the integral is over all space.

The action integral will be given by

$$S = \int dt\ L = \int d^4x\ \ \mathcal{L}\big(\psi(x), \partial_\mu \psi(x)\big)$$

The dynamics of this field will be driven by Hamilton's Principle, viz.

$$\text{if} \quad \psi(x) \to \psi(x) + \delta\psi(x) \quad \text{then} \quad \delta S = 0$$

This will lead to Euler-Lagrange equations

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\{\partial_\mu \psi(x)\}} \right] - \frac{\partial \mathcal{L}}{\partial \psi(x)} = 0$$

If there are many fields $\psi_1(x), \psi_2(x), \dots, \psi_n(x)$ the Lagrangian is

$$L = \int d^3\vec{x} \;\; \mathcal{L}\left(\psi_1(x), \dots, \psi_n(x), \partial_\mu \psi_1(x) \dots, \partial_\mu \psi_n(x)\right)$$

and there are n sets of Euler-Lagrange equations...

The action integral will be given by

$$S = \int dt \;\, L = \int d^4x \;\; \mathcal{L}\left(\psi_1(x), \dots, \psi_n(x), \partial_\mu \psi_1(x) \dots, \partial_\mu \psi_n(x)\right)$$

Nature of field	Euler-Lagrange eqs.	Lagrangian density

Nöther's Theorem (again!)

If, under a transformation $\psi_i(x) \rightarrow \psi_i(x) + \delta\psi_i(x)$, we have $\delta\mathcal{L} = 0$, this will be called a symmetry of the system.

For an infinitesimal change, it follows that

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\{\partial_\mu\psi_i\} + \frac{\partial\mathcal{L}}{\partial\psi_i}\delta\psi_i$$

As before, substitute the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\psi_i} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\right]$$

to get

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\partial_\mu\{\delta\psi_i\} + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\right]\delta\psi_i = \sum_i \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\psi_i\right]$$

i.e.

$$\delta\mathcal{L} = \partial_\mu \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\psi_i = \partial_\mu j^\mu$$

where $j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\psi_i$ is called the Nöther current.

Now, for a symmetry, $\delta\mathcal{L} = 0 \Rightarrow \partial_\mu j^\mu = 0$

i.e. we get an equation of continuity for the Nöther current.

Written out explicitly, the equation of continuity assumes the usual form, i.e.

$$\partial_\mu j^\mu = 0 \;\Rightarrow\; \partial_t j^0 + \vec{\nabla}.\vec{j} = 0$$

Now, integrating over all space,

$$\partial_t \int d^3\vec{x}\; j^0 + \int d^3\vec{x}\; \vec{\nabla}.\vec{j} = 0 \qquad \Rightarrow \partial_t \int d^3\vec{x}\; j^0 + \oint \vec{j}.\hat{n}\, ds = 0$$

i.e. $\partial_t \int d^3\vec{x}\; j^0 = 0$

We define $Q = \int d^3\vec{x}\; j^0$ as the <u>Nöther charge</u>

Emmy Nöther (1882 – 1935) proved her famous theorem in 1915. She was one of the first women to hold an official professorship in a European University – at Göttingen. She did pioneering work on invariants, abstract algebra and topology. In 1933 she moved to the USA, where she died of cancer after two years.

Gauge Invariance of a Complex Scalar Field

The Lagrangian density

$$\mathcal{L} = \partial^{\mu}\varphi^*(x)\partial_{\mu}\varphi(x) - M^2\varphi^*(x)\varphi(x)$$

is manifestly invariant under a global gauge transformation

$$\varphi(x) \rightarrow \varphi'(x) = e^{-ig\theta}\varphi(x)$$

where θ is an arbitrary (real) constant and g is a (real) constant specific to the field...

Also: $\varphi(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)]$ and $\varphi^*(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) - i\varphi_2(x)]$

$$\left.\begin{aligned} \varphi'_1(x) &= \varphi_1(x)\cos g\theta - \varphi_2(x)\sin g\theta \\ \varphi'_2(x) &= \varphi_1(x)\sin g\theta + \varphi_2(x)\cos g\theta \end{aligned}\right\}$$

complex rotation

This set of transformations forms an Abelian (commutative) group

Proof:

Group product $\Rightarrow$ successive transformations $\varphi(x) \rightarrow e^{-ig\theta_2} e^{-ig\theta_1} \varphi(x)$

1. *closure* : $e^{-ig\theta_2} e^{-ig\theta_1} = e^{-ig(\theta_2+\theta_1)}$
2. *associativity* : $e^{-ig\theta_3}(e^{-ig\theta_2} e^{-ig\theta_1}) = (e^{-ig\theta_3} e^{-ig\theta_2}) e^{-ig\theta_1}$
$$= e^{-ig(\theta_3+\theta_2+\theta_1)}$$
3. *identity* : $\theta = 0$; $e^0 = 1$
4. *inverse* : $e^{+ig\theta} e^{-ig\theta} = e^0 = 1$
5. *commutativity*: $e^{-ig\theta_2} e^{-ig\theta_1} = e^{-ig\theta_1} e^{-ig\theta_2} = e^{-ig(\theta_1+\theta_2)}$

This set of phases $e^{-ig\theta}$ forms the group of unitary 1×1 matrices: U(1)

These are global U(1) gauge transformations

Nöther current corresponding to the global U(1) gauge symmetry:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial\{\partial_\mu \varphi\}}\delta\varphi + \frac{\partial \mathcal{L}}{\partial\{\partial_\mu \varphi^*\}}\delta\varphi^*$$

If $\mathcal{L} = \partial^\mu \varphi^*(x)\partial_\mu \varphi(x) - M^2\varphi^*(x)\varphi(x)$, we get

$$\frac{\partial \mathcal{L}}{\partial\{\partial_\mu \varphi\}} = \partial^\mu \varphi^* \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial\{\partial_\mu \varphi^*\}} = \partial^\mu \varphi$$

Now consider an infinitesimal gauge transformation, i.e. $\theta \ll 1$

$$\delta\varphi(x) = \varphi'(x) - \varphi(x) = \left(e^{-ig\theta} - 1\right)\varphi(x) \approx -ig\theta\varphi(x)$$

$$\delta\varphi^*(x) = \varphi'^*(x) - \varphi^*(x) = \left(e^{+ig\theta} - 1\right)\varphi^*(x) \approx +ig\theta\varphi^*(x)$$

Plugging in these values..

$$j^\mu = \partial^\mu \varphi^* [-ig\theta\varphi(x)] + \partial^\mu \varphi[+ig\theta\varphi^*(x)]$$

$$= -ig\theta[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\,]$$

Drop the θ factor:

$$J^\mu = -ig[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\,] = -ig\varphi^* \overleftrightarrow{\partial_\mu} \varphi$$

scalar current

Nöther charge:

$$Q = \int d^3\vec{x}\, j^0 = g \int d^3\vec{x}\, (-i)[\partial^0 \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^0 \varphi\,]$$

This is nothing but the probability for a Klein-Gordon particle,

i.e. gauge symmetry leads to conservation of probability...

Normalisation: $\int d^3\vec{x}\, (-i)[\partial_t\, \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial_t\, \varphi\,] = 1$

i.e. $Q = g$ U(1) charge of $\varphi(x)$

A global gauge transformation is not compatible with relativity

does not account for finite time of signal propagation

Replace it with a local U(1) gauge transformation:

$$\varphi(x) \to \varphi'(x) = e^{-ig\theta(x)}\varphi(x)$$

which also forms a U(1) group

(Set of global U(1) gauge transfns $\subset$ set of local U(1) gauge transfns)

Demand: The action S should be invariant under this transformation, since it is physically meaningful

Under this local U(1) g.t. the fields change to

$$\varphi(x) \rightarrow \varphi'(x) = e^{-ig\theta(x)}\varphi(x)$$
$$\varphi^*(x) \rightarrow \varphi'^*(x) = e^{+ig\theta(x)}\varphi^*(x)$$

The Lagrangian changes to

$$\mathcal{L}' = \partial^\mu \varphi'^*(x)\partial_\mu \varphi'(x) - M^2\varphi'^*(x)\varphi'(x)$$
$$= \partial^\mu\left[e^{+ig\theta(x)}\varphi^*(x)\right]\partial_\mu\left[e^{-ig\theta(x)}\varphi(x)\right] - M^2\varphi^*(x)\varphi(x)$$
$$= \mathcal{L} + ig\partial_\mu\theta\,\partial^\mu(\varphi^* - \varphi) - g^2(\varphi^*\varphi)\,\partial^\mu\theta\,\partial_\mu\theta$$

The theory is no longer gauge invariant!!

This is not physically acceptable, because then we would be able to measure phases in quantum mechanics, which we cannot $\Rightarrow$ paradox

Something must be missing...

Take the Lagrangian density

$$\mathcal{L} = [\partial^\mu \varphi(x)]^* [\partial_\mu \varphi(x)] - M^2 \varphi^*(x) \varphi(x)$$

and rewrite it as

$$\mathcal{L} = [\partial^\mu \varphi(x) + igA^\mu(x)\varphi(x)]^* [\partial_\mu \varphi(x) + igA_\mu(x)\varphi(x)] - M^2 \varphi^*(x) \varphi(x)$$

where $A_\mu(x)$ is a gauge field introduced to get gauge invariance.

Shorter notation: write $\partial_\mu \varphi + igA_\mu \varphi = (\partial_\mu + igA_\mu)\varphi \equiv D_\mu \varphi$

The Lagrangian density becomes

$$\mathcal{L} = [D^\mu \varphi(x)]^* [D_\mu \varphi(x)] - M^2 \varphi^*(x)\varphi(x)$$

Under a local U(1) g.t., we have seen that

$$\varphi(x) \to \varphi'(x) = e^{-ig\theta(x)}\varphi(x)$$

$$D_\mu \varphi(x) \to D'_\mu \varphi'(x) = e^{-ig\theta(x)} D_\mu \varphi(x)$$

so that the Lagrangian density becomes trivially invariant.

The construction $D_\mu \varphi$ transforms in the same way as the $\varphi(x)$, so we call it a covariant derivative.

Hermann Weyl (1885 – 1955) – pioneer of group theory in physics

Vladimir Fock (1898 – 1974) – pioneer of quantum field theory

Fritz London (1900 – 1954) – pioneer of quantum many-body systems

1918

1927

Write out the Lagrangian density in full...

Do we understand this A_μ field?

Consider its Euler-Lagrange equation: $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \{\partial_\mu A_\nu\}} \right] - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$$gJ_\nu + g^2 \varphi^* \varphi \, A_\nu = 0 \quad \Rightarrow \quad A_\nu = \frac{J_\nu}{g\varphi^*\varphi} = \frac{1}{g} \frac{\varphi^* \overleftrightarrow{\partial}_\mu \varphi}{\varphi^* \varphi}$$

$\Rightarrow$ nonlinear Lagrangian... nonlinear wave equations... no quantum theory

Again, something must be missing....

The A_ν fields must have some dynamics,

i.e. there must be a term with $\partial_\mu A_\nu$

This term must be both Lorentz-invariant and gauge-invariant

Under a local U(1) g.t., we know that $A_\nu \to A_\nu + \partial_\nu \theta$

Then, $\partial_\mu A_\nu \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \theta$

and $\partial_\nu A_\mu \to \partial_\nu A_\mu + \partial_\nu \partial_\mu \theta$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$ field strength tensor

Lorentz-invariant construction: $F_{\mu\nu} F^{\mu\nu}$

Full Lagrangian:

$$\mathcal{L} = (\partial^\mu \varphi)^* \, \partial_\mu \varphi - M^2 \varphi^* \varphi + ig\left(\varphi^* \overleftrightarrow{\partial}_\mu \varphi\right) A^\mu + g^2 \varphi^* \varphi \, A^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The Euler-Lagrange equation becomes:

$$\partial_\mu F^{\mu\nu} = gJ^\nu - g^2 \varphi^* \varphi \, A^\nu$$

For small g, this reduces to

$$\partial_\mu F^{\mu\nu} = gJ^\nu$$

i.e. identical with Maxwell's equations...

It follows that the A_μ must be the electromagnetic field and $g = qe$.

The quantum mechanics of a complex scalar field has no physical meaning unless we couple it to an electromagnetic field...

electromagnetism $\Leftrightarrow$ inability to measure phase of a wavefunction

Gauge Invariance of a Dirac Field

The Lagrangian density

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\partial_{\mu}\psi - m\,\bar{\psi}\psi$$

is manifestly invariant under a global U(1) gauge transformation

$$\psi(x) \rightarrow \psi'(x) = e^{-ie\theta}\,\psi(x)$$

where θ is an arbitrary (real) constant and e is a (real) constant specific to the field...

Easy to show that the Nöther current corresponding to this symmetry is the Dirac current $J^{\mu} = e\,\overline{\psi}\,\gamma^{\mu}\psi$ and the Nöther charge is just

$$Q = \int d^3\vec{x}\; j^0 = e\int d^3\vec{x}\;\bar{\psi}\,\gamma^0\psi = e\int d^3\vec{x}\,\psi^{\dagger}\psi = e$$

For local U(1) gauge invariance, replace $\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi$ by

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,D_{\mu}\psi - m\,\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$$

where, as before, $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

We will also get Maxwells' equations: $\partial_{\mu}F^{\mu\nu} = eJ^{\nu}$

Quantum Electrodynamics (QED)

Once we have Maxwell's equations, we can write

.

Choose the Lorentz gauge $\partial_\mu A^\mu = 0$ and we recover

$$\Box A^\nu = eJ^\nu$$

In static limit, this leads to Coulomb's law and a long-range interaction

Can the photon have a mass?

Then we would have a Klein-Gordon equation: $\left(\Box + M_\gamma^2\right) A^\nu = eJ^\nu$

coming from a Maxwell equation: $\partial_\mu F^{\mu\nu} + M_\gamma^2 A^\nu = eJ^\nu$

If this is the Euler-Lagrange equation, the Lagrangian density must have an extra *mass term*

$$\mathcal{L}_M = \tfrac{1}{2} M_\gamma^2 A^\nu A_\nu$$

Under a gauge transformation, $A_\nu \to A_\nu + \partial_\nu \theta$, and it follows that

$$\mathcal{L}_M \to \tfrac{1}{2} M_\gamma^2 (A^\nu + \partial^\nu \theta)(A_\nu + \partial_\nu \theta) \neq \tfrac{1}{2} M_\gamma^2 A^\nu A_\nu$$

For gauge invariance, we must set $M_\gamma = 0$, i.e. the photon must be massless

gauge invariance ⇔ long range interactions