Electroweak Unification and the Standard Model

Lecture 3

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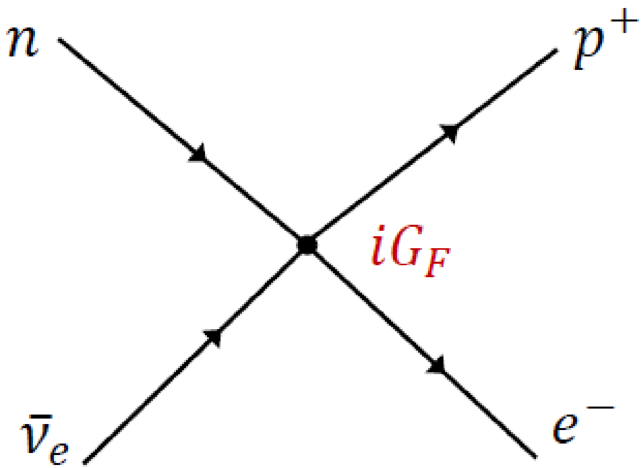
Fermi's theory of beta decay

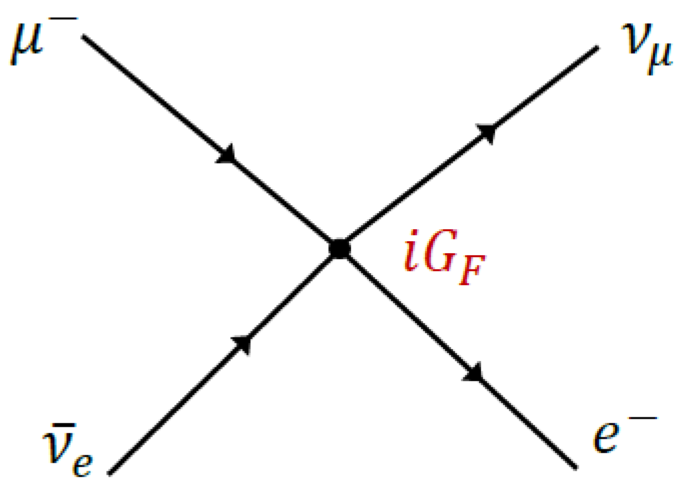


Weak interaction Hamiltonian:

$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}} \, \bar{p}\gamma^{\mu} n \;\; \bar{e}\gamma_{\mu}\nu_e$$

dimension of G_F is M^{-2} : Fermi coupling constant

Fermi's guess: universality of weak interactions



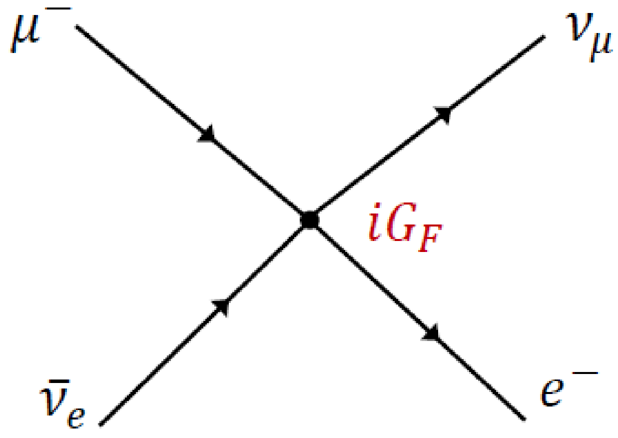
Use this to calculate the muon lifetime:

$$\tau_\mu \approx \frac{192\,\pi^3}{G_F^2\, M_\mu^5} \approx 2.25 \times 10^{-6}\ \mathrm{s}$$

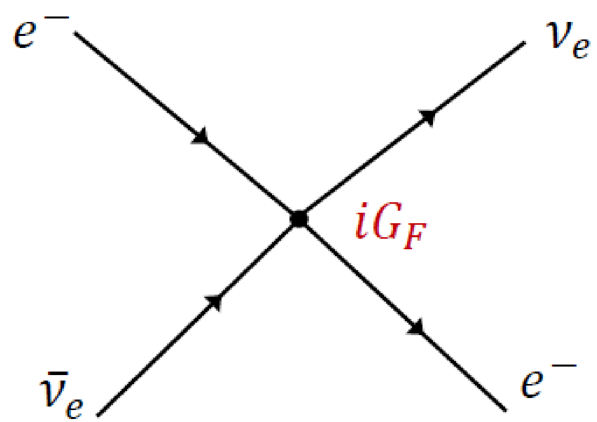
Spectacular agreement with the experimental value 2.197×10^{-6} s

Weak scattering processes: the unitarity problem

If we have a vertex



as Fermi postulated, then, by universality, we should also have



and it should be possible to have a scattering process

$$e^- + \nu_e \to e^- + \nu_e$$

Cross-section:

$$\sigma \approx \frac{G_F^2}{\pi} s \left(1 - \frac{M_e^2}{s}\right)$$

where $s = \left(p_e + p_{\nu_e}\right)^2 = E_{\rm cm}^2$.

Clearly, as $s \uparrow$, $\sigma \uparrow$ ...

unitarity violation

S-matrix

The S-matrix is a quantum mechanical operator which connects the initial state to the final state.

$$|f\rangle = \hat{\mathcal{S}}|i\rangle$$

Probability amplitude for a scattering process is $\mathcal{S}_{fi} = \langle i|f\rangle = \langle i|\hat{\mathcal{S}}|i\rangle$

S-matrix may be expanded perturbatively as $\mathcal{S} = \mathbb{1} + ig\mathcal{T}$ which means $\mathcal{S}^\dagger = \mathbb{1} - ig\mathcal{T}^\dagger$

Since the S-matrix connects the initial state to the final state, it must be unitary

$$\mathcal{S}^\dagger\mathcal{S} = \mathbb{1} + ig\cancel{(\mathcal{T} - \mathcal{T}^\dagger)} + \cancel{g^2\mathcal{T}^\dagger\mathcal{T}} = \mathbb{1}$$

i.e. unitarity is maintained so long as $\mathcal{T} = \mathcal{T}^\dagger$ and $|\langle g\mathcal{T}\rangle| \ll 1$

Now the cross-section, by a dimensional argument, will vary as $\sigma \sim \frac{1}{s}\int d\Phi\,|\langle g\mathcal{T}\rangle|^2$

If this increases faster than $(\ln s)^2$ then $|\langle g\mathcal{T}\rangle|$ will keep increasing and at some energy It will violate $|\langle g\mathcal{T}\rangle| \ll 1$ i.e. unitarity

Froissart bound

Cross-section:

$$\sigma \approx \frac{G_F^2}{\pi} s \left(1 - \frac{M_e^2}{s}\right)$$

where $s = (p_e + p_{\nu_e})^2 = E_{\text{cm}}^2$.

Clearly, as $s \uparrow$, $\sigma \uparrow$ ...

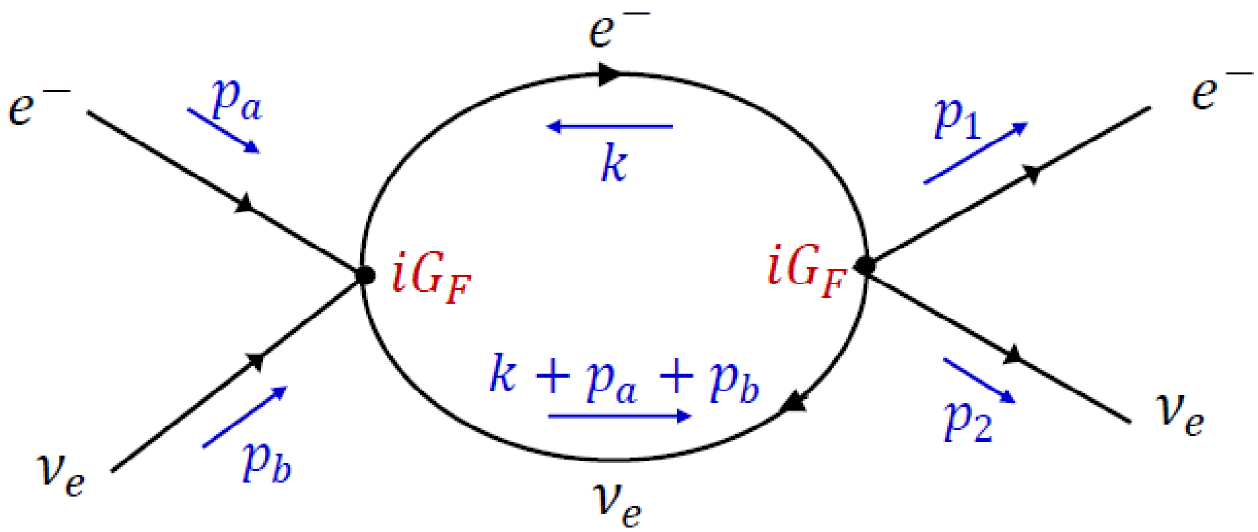
unitarity violation

Perhaps this arises because we took only the LO diagram... ?

... inclusion of higher orders may soften the growth with energy...

...but this leads to a new problem: renormalisability

Consider the simplest one-loop contribution to $e^-\nu_e \to e^-\nu_e$:



The effective coupling due to this would be

$$\frac{iG_F}{\sqrt{2}} \to \frac{iG_F}{\sqrt{2}} + \frac{(iG_F)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - M_e} \frac{1}{\not{k} + \not{p}_a + \not{p}_b}$$

Since k is integrated over all values, the dominant contribution will come from $k \to \infty$, i.e.

$$\frac{iG_F}{\sqrt{2}} \rightarrow \frac{iG_F}{\sqrt{2}} + \frac{(iG_F)^2}{2} \int_0^\infty \frac{2\pi^2 k^3 dk}{(2\pi)^4} \frac{1}{\cancel{kk}}$$

This extra contribution is quadratically divergent, i.e. if we put a momentum cutoff $k \leq \Lambda$ then,

$$\frac{iG_F}{\sqrt{2}} \rightarrow \frac{iG_F}{\sqrt{2}} + \frac{(iG_F)^2}{16\pi^2} \int_0^\Lambda k\, dk$$

$$= \frac{iG_F}{\sqrt{2}} + \frac{(iG_F)^2}{32\pi^2} \Lambda^2$$

If the NLO contribution ≫ LO contribution, perturbation theory fails...

Such problems arise in QED as well for e, but there the divergences are logarithmic, i.e. proportional to $\log \Lambda$. Moreover, in every order (NLO, NNLO, NNNLO, ...) we always get a similar logarithmic divergence.

These can be summed up, and the result absorbed into the definition of e -- this process is called renormalisation

In the Fermi theory, however, higher and higher powers of Λ^2 keep coming with higher and higher orders, and there is no scope for renormalisation...

Does this mean that the Fermi theory is wrong?

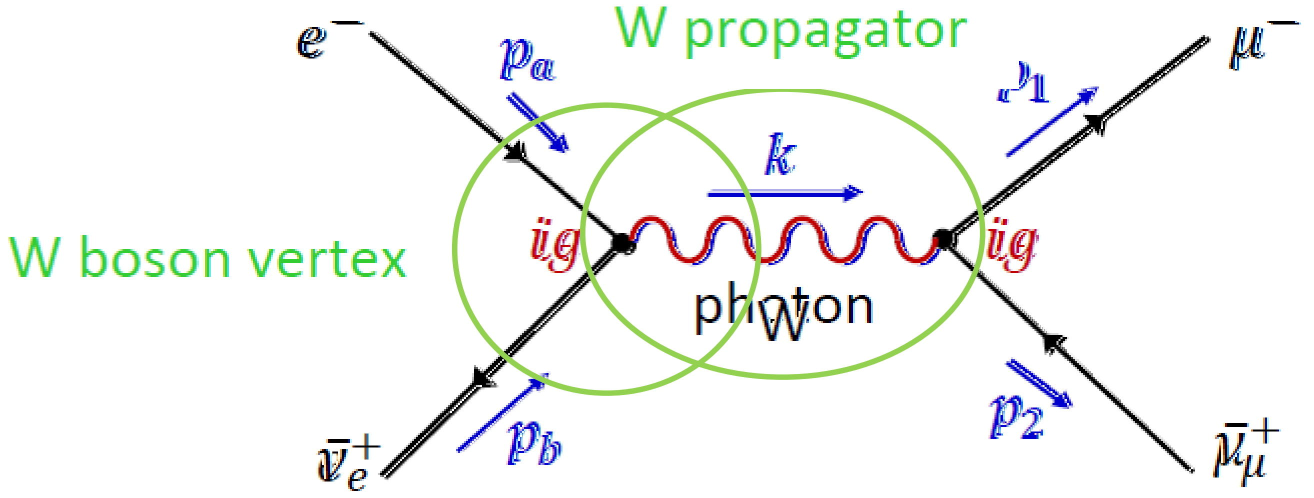
Correspondence Principle: every new theory should reduce to the old theory in the range of parameters where that theory was successful

Fermi theory must be a low-energy effective theory...

Intermediate Vector Bosons (IVB):

Schwinger (1953): if renormalisation is possible in QED, can we make it possible in weak interactions by copying the same form?

Consider the following weak process: $e^- + \bar{\nu}_e \rightarrow \mu^- + \bar{\nu}_\mu$



Taking Fermi's idea a step further...

$$\sigma \approx \frac{G_F^2}{\pi} s \left(1 - \frac{M_e^2}{s}\right)$$

$$i\mathcal{M} = \bar{v}(p_b)\, ig\gamma^\mu u(p_a) \frac{-ig_{\mu\nu}}{k^2} \bar{u}(p_1)\, ig\gamma^\nu v(p_2)$$

$$= \frac{ig^2}{k^2} \bar{v}(p_b)\, \gamma^\mu u(p_a)\;\; \bar{u}(p_1)\, \gamma_\mu v(p_2)$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{k^2} = \frac{\boldsymbol{g^2}}{\boldsymbol{s}}$$

Objection: The Fermi coupling constant does not show significant variation with energy as $k^2 \to 0$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{k^2}$$

Schwinger's solution: make the W boson massive

$$\frac{-ig_{\mu\nu}}{k^2} \to \frac{-ig_{\mu\nu} + k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$$

$$i\mathcal{M} = \bar{v}(p_b)\, ig\gamma^\mu u(p_a)\, \frac{-ig_{\mu\nu} + k_\mu k_\nu / M_W^2}{k^2 - M_W^2}\, \bar{u}(p_1)\, ig\gamma^\nu v(p_2)$$

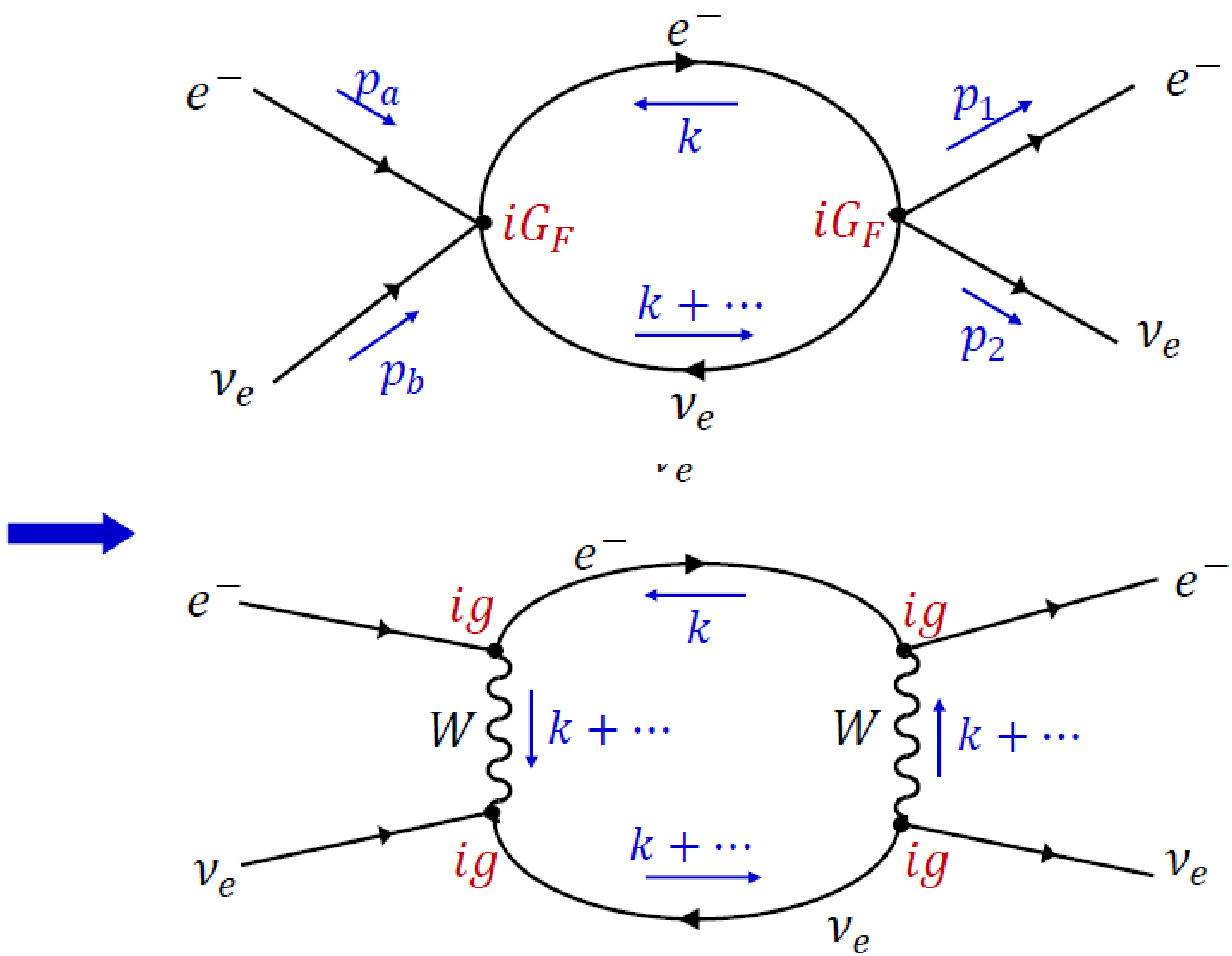
$$= \bar{v}(p_b)\, ig\gamma^\mu u(p_a)\, \frac{-ig_{\mu\nu}}{k^2 - M_W^2}\, \bar{u}(p_1)\, ig\gamma^\nu v(p_2)$$

$$= \frac{ig^2}{k^2 - M_W^2}\, \bar{v}(p_b)\, \gamma^\mu u(p_a)\, \bar{u}(p_1)\, \gamma_\mu v(p_2)$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{k^2 - M_W^2}$$

In the low energy limit, $k^2 \to 0$ we get: $\frac{G_F}{\sqrt{2}} = -\frac{g^2}{M_W^2}$ constant!!

Q. How does this help?



Rewrite the loop integral...

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{\cancel{k}} \frac{1}{\cancel{k}+\cdots} \rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{1}{\cancel{k}} \frac{1}{k^2 - M_W^2} \frac{1}{\cancel{k}+\cdots} \frac{1}{k^2 - M_W^2}$$

$$\propto \int k^3 dk \; \frac{1}{k^2} \; \frac{1}{k^4}$$

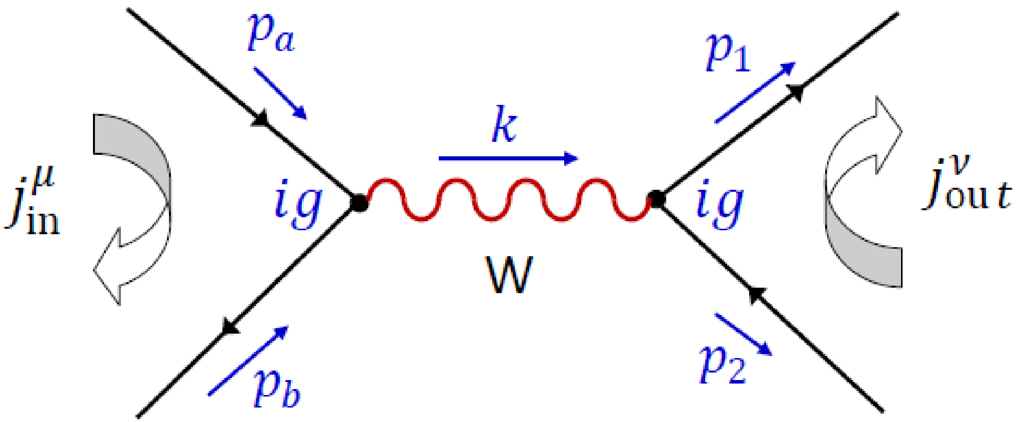
$$\propto \int \frac{dk}{k^3} \qquad \text{finite!}$$

As it would be in QED...

But we have cheated... $\frac{-ig_{\mu\nu}}{k^2} \rightarrow \frac{-ig_{\mu\nu} + k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{\cancel{k}} \frac{k_\mu k_\nu}{k^2 - M_W^2} \frac{1}{\cancel{k}+\cdots} \frac{k_\mu k_\nu}{k^2 - M_W^2} \propto \int k^3 dk \; \frac{1}{k^2} \; \frac{k^4}{k^4} \propto \int k dk$$

What does the propagator couple to?



$$\partial_\mu j^{\mu}_{\text{in}}(x) = 0$$

$$\mathcal{M} \propto j^{\mu}_{\text{in}} \, ig \, \frac{-ig_{\mu\nu} + k_\mu k_\nu / M_W^2}{k^2 - M_W^2} \, ig \, j^{\nu}_{\text{out}}$$

The offending term will go away if $j^{\mu}_{\text{in}} k_\mu = 0$ and/or $k_\nu j^{\nu}_{\text{out}} = 0$

To have conserved currents, there must be a gauge symmetry...

Julian Schwinger (1918 – 1994) was a pioneer of quantum field theory and developed the idea of correlation functions to study interacting fields. He was also the first to realize that neutrinos have more than one flavour.

To have conserved currents, there must be a gauge symmetry...

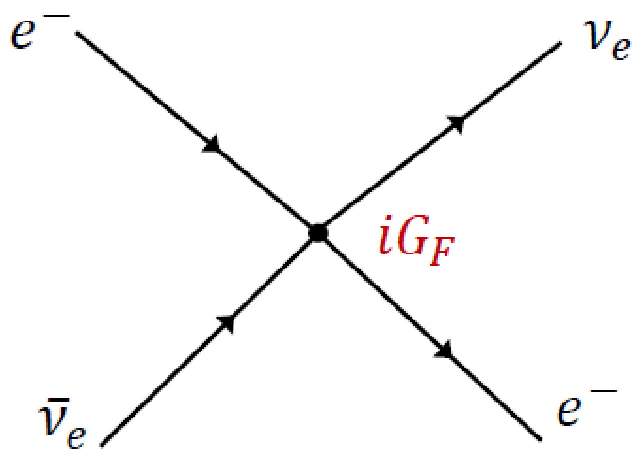
But this cannot be a U(1) gauge symmetry, like QED

Why not? Because the W boson is charged,
i.e. there are two W bosons

$$W_\mu^+ = \frac{1}{\sqrt{2}}(W_1^+ + iW_2^+) \qquad W_\mu^- = \frac{1}{\sqrt{2}}(W_1^+ - iW_2^+)$$

i.e. the group of gauge symmetries must have at least <u>two</u> generators

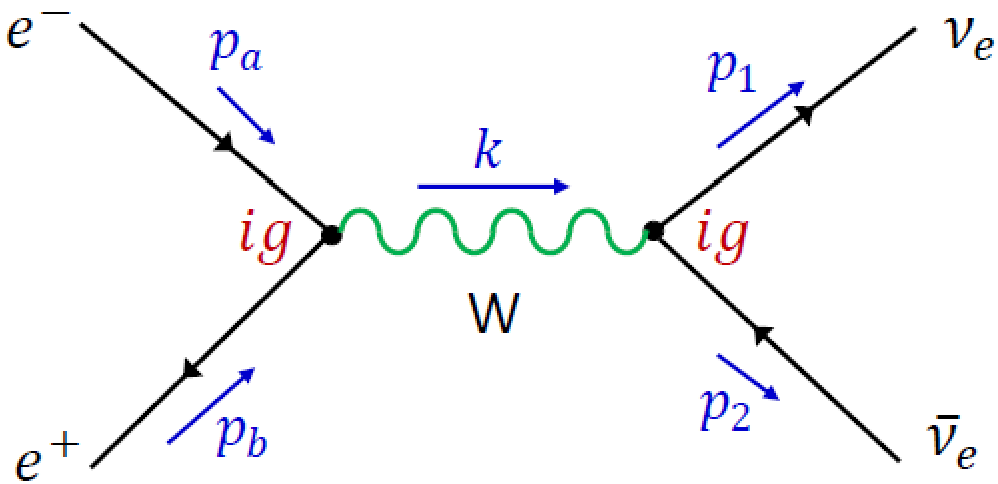
In fact, if we have a four-fermion theory with the vertex



there is nothing, in principle, to prevent a process like

$$e^- + e^+ \to \nu_e + \bar{\nu}_e$$

How will this look in the IVB theory?



So, perhaps we have a neutral W boson also – a W_μ^0

This W_μ^0 cannot be the photon because it couples to neutrinos...

i.e. the group of gauge symmetries must have three generators

after U(1), the next unitary group is SU(2), which has 3 generators...

The matrices of SU(2) don't commute, so it is a non-Abelian group...

Nonabelian Gauge Theory

Consider a scalar multiplet $\Phi(x)$ of length n, i.e.

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \vdots \\ \varphi_n(x) \end{pmatrix}$$

where each $\varphi_i(x)$ $(i = 1, \dots, n)$ is a complex scalar field.

Construct the 'free' Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi$$

This is just a shorthand for n mass-degenerate free scalar fields, i.e.

$$\mathcal{L} = \sum_{i=1}^{n} \left(\partial^\mu \varphi_i^* \, \partial_\mu \varphi_i - M^2 \varphi_i^* \varphi_i \right)$$

Now consider a global SU(N) gauge transformation

$$\Phi(x) \to \Phi^{'}(x) = \mathbb{U}\Phi(x)$$

where $\mathbb{U}$ is a SU(N) matrix, i.e. $\mathbb{U}^{\dagger}\,\mathbb{U} = \mathbb{1}$ and $\det \mathbb{U} = \ +1$, where

$$\mathbb{U} = \begin{pmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & & \vdots \\ U_{n1} & \cdots & U_{nn} \end{pmatrix}$$

n and N are different (in general)

$$n \geq N$$

If equal it is the fundamental representation

The number of free (real) parameters in this SU(N) matrix is

$$p = 2N^2 - N - 2^N C_2 - 1 = N^2 - 1$$

We can write this SU(N) transformation in the form $\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}}$

where the $\vec{\theta} = \left(\theta_1, \cdots, \theta_p\right)$ are free (real) parameters

and the $\vec{\mathbb{T}} = \left(\mathbb{T}_1, \cdots, \mathbb{T}_p\right)$ are the generators of SU(N)

$$\vec{\theta}.\vec{\mathbb{T}} = \sum_{a=1}^{p} \theta_a \mathbb{T}_a$$

Under this gauge transformation

$$\Phi(x) \to \Phi'(x) = \mathbb{U}\,\Phi(x)$$
$$\Phi^\dagger(x) \to \Phi'^\dagger(x) = \Phi^\dagger(x)\,\mathbb{U}^\dagger$$

.

Thus, this system of n mass-degenerate free scalar fields possesses a SU(N) global gauge symmetry — with p conserved currents/charges.

The next step is to convert this to a SU(N) local gauge symmetry, i.e.

$$\Phi(x) \to \Phi'(x) = \mathbb{U}(x)\,\Phi(x)$$
$$\Phi^\dagger(x) \to \Phi'^\dagger(x) = \Phi^\dagger(x)\,\mathbb{U}^\dagger(x)$$

As in the nonAbelian case, the Lagrangian density will no longer remain gauge invariant...

<u>Solution:</u> define a covariant derivative $\mathbb{D}_\mu = \mathbb{1}\partial_\mu + ig\mathbb{A}_\mu(x)$ where the $\mathbb{A}_\mu(x)$ is a $n \times n$ matrix of gauge fields, i.e.

$$\mathbb{A}^\mu = \begin{pmatrix} a_{11}^\mu & \cdots & a_{1n}^\mu \\ \vdots & & \vdots \\ a_{n1}^\mu & \cdots & a_{nn}^\mu \end{pmatrix}$$

Not all of these need to be independent… ($\mathbb{A}^\mu$ is Hermitian…)

We require the covariant derivative $\mathbb{D}_\mu\Phi$ to transform exactly like Φ, i.e.

$$\mathbb{D}_\mu\Phi \to \mathbb{D}'_\mu\Phi' = \mathbb{U}\,\mathbb{D}_\mu\Phi$$

for then, if we rewrite the Lagrangian density as

$$\mathcal{L} = (\mathbb{D}^\mu\Phi)^\dagger\mathbb{D}_\mu\Phi - M^2\Phi^\dagger\Phi$$

it will be trivially gauge invariant.

How do we ensure that $\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \mathbb{U}\, \mathbb{D}_\mu \Phi$?

By adjusting the transformation of the gauge field matrix $\mathbb{A}^\mu$ ...

If this is to be the same as

$$\mathbb{D}_\mu \Phi = \left(\mathbb{1}\partial_\mu + ig\mathbb{A}_\mu\right)\Phi$$

we must have $ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} + \mathbb{U}^\dagger \partial_\mu \mathbb{U}$

Rewrite

$$ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger\mathbb{A}'_\mu\mathbb{U} + \mathbb{U}^\dagger\partial_\mu\mathbb{U}$$

Quick check: suppose $N = 1$ and $n = 1$, i.e. U(1) gauge symmetry

Then $\mathbb{U} = e^{-ig\theta}$ and $\mathbb{A}_\mu = A_\mu$.

How many independent fields do we require in the $\mathbb{A}_\mu$ matrix?

One can now work out the transformation properties of the $A_a^\mu(x)$ fields in terms of the parameters $\vec{\theta} = (\theta_1, \cdots, \theta_p)$.

(Will do this for specific cases...)

We can also use this expression

$$\mathbb{A}^{\mu}(x) = \sum_{a=1}^{p} A_a^{\mu}(x)\, \mathbb{T}_a = \overrightarrow{A^{\mu}} \cdot \overrightarrow{\mathbb{T}}$$

to write out the interaction terms in the Lagrangian density...

We should complete the Lagrangian density by adding a kinetic term for the gauge fields...

$$\mathbb{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbb{D}_\mu, \mathbb{D}_\nu\right] = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]$$

Now, we have

$$\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \mathbb{U}\ \mathbb{D}_\mu \Phi$$

Thus,

$$\mathbb{F}_{\mu\nu} \to \mathbb{F}'_{\mu\nu} = -\frac{i}{g}\left[\mathbb{D}'_\mu, \mathbb{D}'_\nu\right] = -\frac{i}{g}\left[\mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger, \mathbb{U}\ \mathbb{D}_\nu \mathbb{U}^\dagger\right] = \mathbb{U}\ \mathbb{F}_{\mu\nu}\,\mathbb{U}^\dagger$$

To get gauge invariance, we have to take the trace...

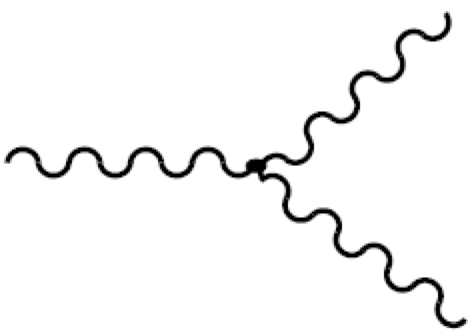
The full Lagrangian density is now

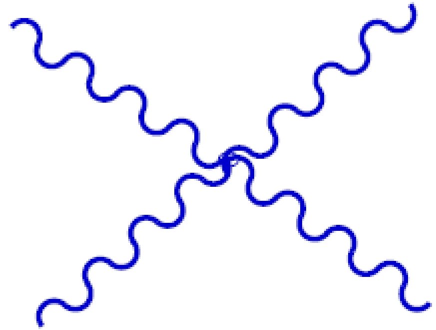
$$\mathcal{L} = (\mathbb{D}^{\mu}\Phi)^{\dagger}\mathbb{D}_{\mu}\Phi - M^2\Phi^{\dagger}\Phi - \tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right]$$

Since $\mathbb{F}_{\mu\nu} = \left(\partial_{\mu}\mathbb{A}_{\nu} - \partial_{\nu}\mathbb{A}_{\mu}\right) + ig\left[\mathbb{A}_{\mu}, \mathbb{A}_{\nu}\right]$

$$\mathbb{F}^{\mu\nu} = (\partial^{\mu}\mathbb{A}^{\nu} - \partial^{\nu}\mathbb{A}^{\mu}) + ig[\mathbb{A}^{\mu}, \mathbb{A}^{\nu}]$$

Leads to triple gauge vertices and quadruple gauge vertices





absent in an Abelian gauge theory, e.g. QED

SU(2) Gauge Theory

Recall that for weak interactions we needed three gauge bosons, the

$$W_\mu^+, W_\mu^-, W_\mu^0$$

This seems to indicate a gauge theory with three generators

and the obvious one to take is an SU(2) gauge theory.

All of the above formalism will work, except that now we must take the generators as

$$\mathbb{T}_1 = \tfrac{1}{2}\sigma_1 \qquad \mathbb{T}_2 = \tfrac{1}{2}\sigma_2 \qquad \mathbb{T}_3 = \tfrac{1}{2}\sigma_3$$

obeying the Lie algebra

$$[\mathbb{T}_a, \mathbb{T}_b] = i\varepsilon_{abc}\,\mathbb{T}_c$$

The full Lagrangian for this is

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi + ig\left[(\partial^\mu \Phi)^\dagger \mathbb{A}_\mu \Phi - \Phi^\dagger \mathbb{A}^\mu \partial_\mu \Phi\right]$$

$$+g^2 \Phi^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \Phi - \tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}\right]$$

$$\Phi = \begin{pmatrix} \varphi_\mathrm{A} \\ \varphi_\mathrm{B} \end{pmatrix}$$

where

$$\mathbb{A}^\mu = A_1^\mu \mathbb{T}_1 + A_2^\mu \mathbb{T}_2 + A_3^\mu \mathbb{T}_3$$

We can also expand

$$\mathbb{F}^{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]$$

$$= F_1^{\mu\nu} \mathbb{T}_1 + F_2^{\mu\nu} \mathbb{T}_2 + F_3^{\mu\nu} \mathbb{T}_3$$

where

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g\varepsilon_{abc} A_b^\mu A_c^\nu$$