Statistical errors, Confidence intervals and limits

We have seen ~~ave~~ an example of error on parameters as the standard ~~definit~~ deviation of the distribution of estimated lifetime - $\hat{\tau}$.

Let us now inspect the meaning of quoted errors more closely.

Let → θ be a parameter

$\hat{\theta}$ estimate of $\hat{\theta}$.

$g(\hat{\theta})$ distribution of $\hat{\theta}$.

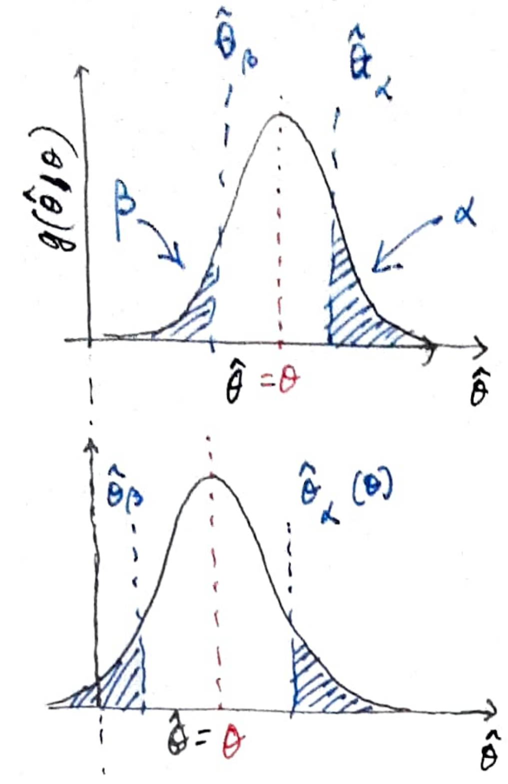
$\sigma_{\hat{\theta}}$ standard deviation of $g(\hat{\theta})$

$\hat{\sigma}_{\hat{\theta}}$ estimate of standard deviation of $\hat{\theta}$.

Then: $\hat{\theta}_{obs} \pm \hat{\sigma}_{\hat{\theta}}$ repeated estimates of θ, all based on sample size n, the quoted error is an estimate the standargd deviation of $\hat{\theta}$, while $\hat{\theta}_{obs}$ is the estimated value of the parameter θ in that experiment, of sample size n, that is being reported. If $g(\hat{\theta})$ is Gaussian, this is the so-called 68.3% confidence interval

What can we infer about the unknown true parameter θ from this measurement $\hat{\theta}_{obs} \pm \hat{\sigma}_{\hat{\theta}}$?

Classical confidence intervals (Neyman-construction)



Given θ:

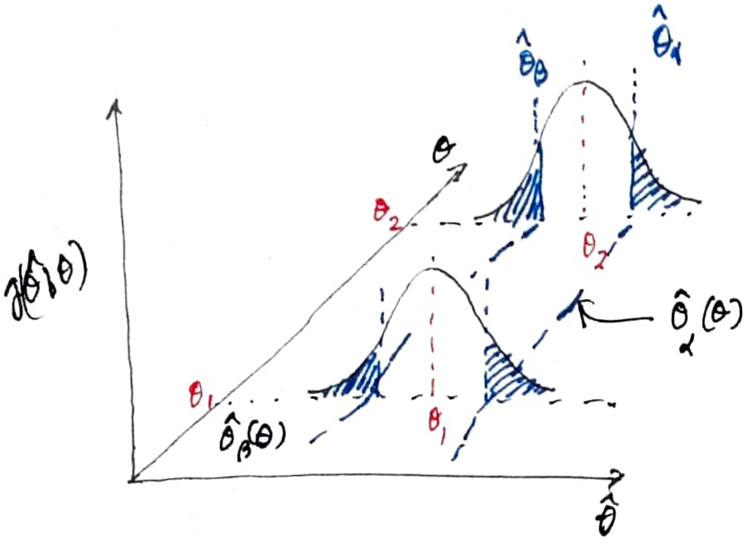
α is the prob of finding $\hat\theta \geq \hat\theta_\alpha$

β " " " " " $\hat\theta \leq \hat\theta_\beta$

for another hypothesised $\hat\theta$ the distribution will shift.

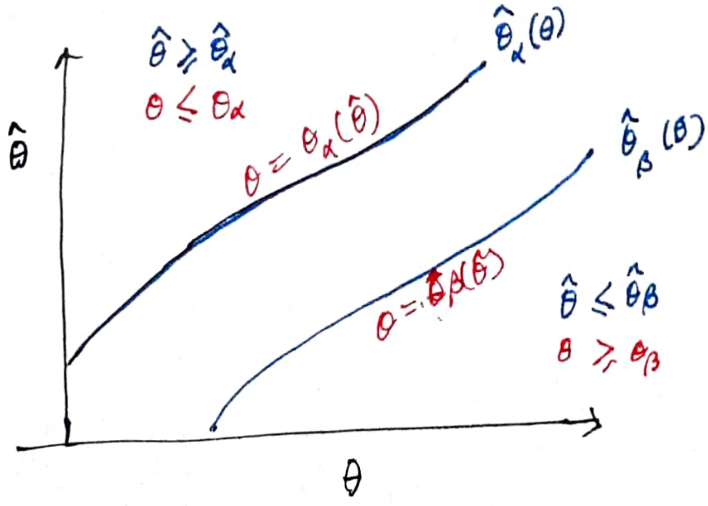
$\Rightarrow$ $\hat\theta_\alpha$, $\hat\theta_\beta$ will shift.

$\Rightarrow$ $\hat\theta_\alpha(\theta)$, $\hat\theta_\beta(\theta)$ are functions of θ.



The blue dashed line (for one parameter) is confidence belt. It is a set of ~~interval~~ acceptance regions $[\hat\theta_\beta, \hat\theta_\alpha]$

$$P(\hat\theta_\beta \leq \hat\theta \leq \hat\theta_\alpha) = 1-\alpha-\beta.$$



We can invert the $\hat\theta_\alpha$ and $\hat\theta_\beta$ fns.

Think θ as y axis and $\hat\theta$ as x axis

Then $\theta = \theta_\alpha(\hat\theta) = \hat\theta_\alpha^{-1}(\theta)$

$\theta = \theta_\alpha(\hat\theta) = \hat\theta_\beta^{-1}(\theta)$

are the inverted functions.

Note that $\cancel{\hat{\theta}_\alpha \geq}$ $\hat{\theta} \geq \hat{\theta}_\alpha \Rightarrow \theta \leq \theta_\alpha$

$\hat{\theta} \leq \hat{\theta}_\beta \Rightarrow \theta \geq \theta_\beta$

Therefore $P(\hat{\theta} \geq \hat{\theta}_\alpha) = \alpha \Rightarrow P(\theta \leq \theta_\alpha) = \alpha$

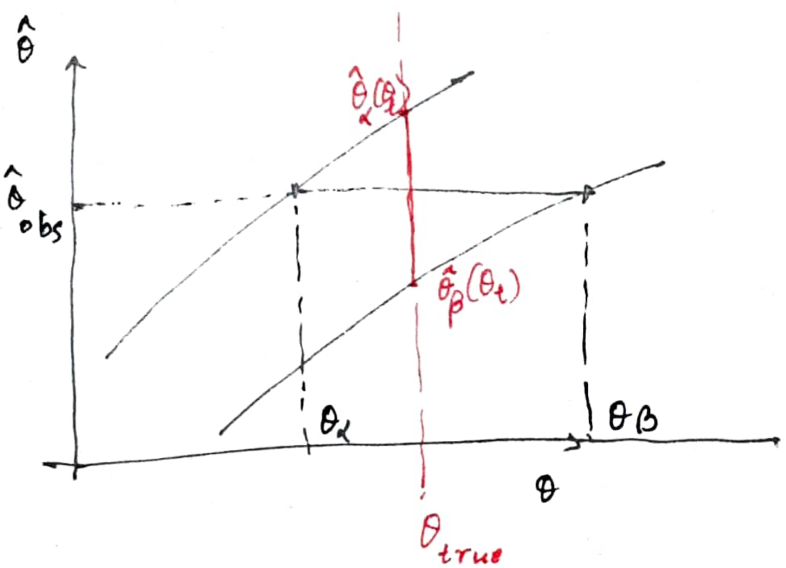
$\cancel{P(\hat{\theta} \geq \hat{\theta}_\beta}$

$P(\hat{\theta} \leq \hat{\theta}_\beta) = \beta \Rightarrow P(\theta \geq \theta_\beta) = \beta$

Taken together

$$P(\theta_\alpha \leq \theta \leq \theta_\beta) = 1 - \alpha - \beta.$$

The interval $[\theta_\alpha, \theta_\beta]$ for an observation $\hat{\theta}_{obs}$ is called the confidence interval.



Probability that θ_t will lie betw. $[\theta_\alpha, \theta_\beta]$ is $1-\alpha-\beta$.

One can see that

if $\hat{\theta}_{obs}$ is inside the interval $[\hat{\theta}_\alpha(\theta_t), \hat{\theta}_\beta(\theta_t)]$ then $[\theta_\alpha, \theta_\beta]$ covers the true value θ_t

Feldman Cousins unified approach for constructing classical interval.

a confidence interval is a member of a set satisfying

$$P(\mu_1 \leq \mu \leq \mu_2) = \alpha \qquad — ①$$

Here μ_1 and μ_2 are ~~rand~~ functions of measured x.

$\{[\mu_1, \mu_2]\}$ from an ensemble of experiments.

* Different from the Baysian statement that degree of belief that μ_t is in $[\mu_1, \mu_2]$ is α.

If ① is satisfied → intervals cover μ at c.L. α.

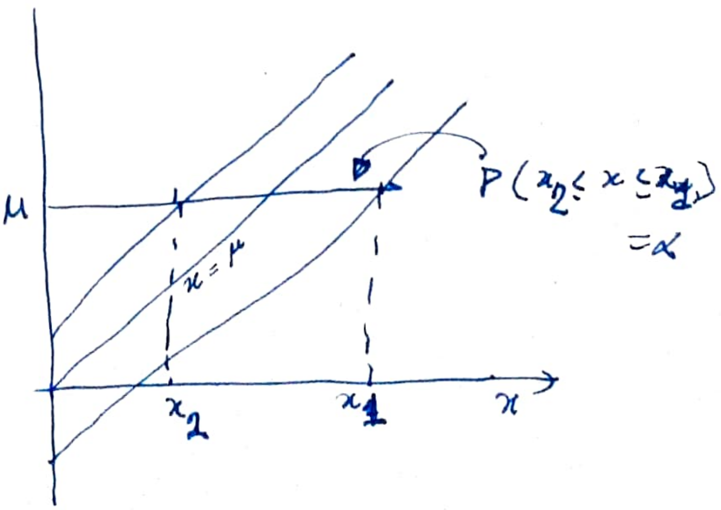
$P(\mu_1 \leq \mu \leq \mu_2) < \alpha$ → under coverage

$P(\mu_1 \leq \mu \leq \mu_2) > \alpha$ → over coverage.

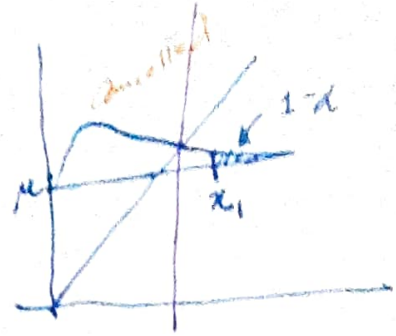
A set $[\mu_1, \mu_2]$ is conservative if it overcovers for some value of μ but never undercovers.
↳ loss of ~~sensitivity~~ power ~~to~~ in hypothesis testing.

There is freedom of choice in the horizontal interval $[x_1, x_2]$

→ needs auxiliary ~~conditions~~ criteria



Popular choices:



1) $P(x < x_1 | \mu) = 1-\alpha$

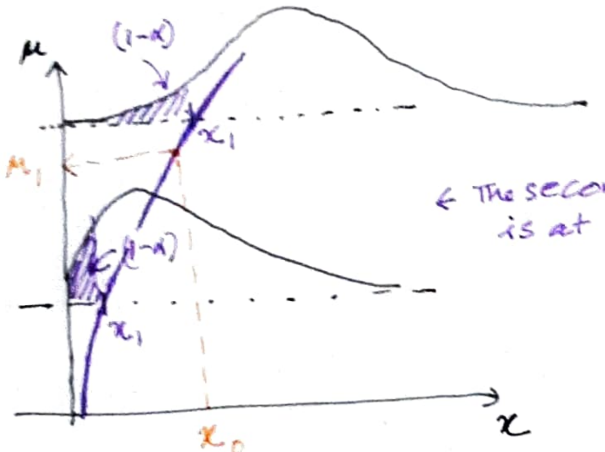
: upper confidence limits
(e.g. x-sections from non-observance.)

This satisfies $P(\mu > \mu_2) = 1-\alpha$

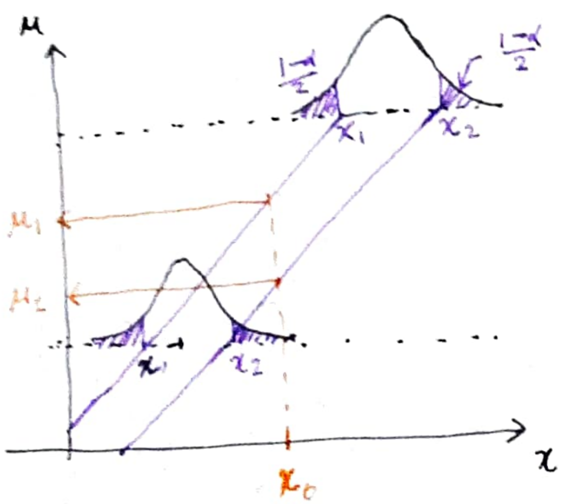
2) $P(x < x_1 | \mu) = P(x > x_2 | \mu) = (1-\alpha)/2$

central interval. This satisfies P

Example of 1. number counting experiment, Poisson distributed.



Example 2 measurement of mass.



Calculate the probability of observing ~~a~~ zero events ($n=0$) for the cases:

		Ans	formula
(i)	$\mu = 0.5,\ b = 3.0$	0.03	$\dfrac{e^{-(\mu+b)}(\mu+b)^n}{n!}$
(ii)	$\mu = 0.0,\ b = 3.0$	0.05	

Intervals from ~~based on~~ an ordering principle based on ~~derived from~~ likelihood ratios.

Take the exercise above. If we tried getting the best fit value of μ from data, obviously

$$n = b + \mu \quad \Rightarrow \quad \mu_{best} = n - b$$

$$\therefore \quad P(n \mid \mu_{best}) = \frac{e^{-n}\, n^n}{n!} \quad \text{for } n > b. \quad \text{else} \quad \frac{e^{-b}\, b^n}{n!}$$

Define

$$R = \frac{P(n/\mu)}{P(n/\mu_{best})} = \frac{e^{-(\mu+b)}\,(\mu+b)^n}{e^{-n}\quad n^n}$$

$$= e^{n-(\mu+b)} \left(\frac{\mu+b}{n}\right)^n, \quad n \geq b.$$

for $n \leq b$ $\quad \mu_{best} = 0. \quad \Rightarrow R = e^{-\mu} \cdot \left(\frac{\mu+b}{b}\right)^n.$

~~Calculate~~ Write a code to compute the values of R and check if it matches with table-1 of the paper by Feldman and Cousins, for case (i) of previous problem, i.e. $\mu = 0.5, b = 3.0$.
Note: μ can not be negative.

Once the rank is obtained values of ~~R are~~ n are ~~added to~~ included in the acceptance region in decreasing order of R. until $\sum_n P(n \mid \mu) \geq \alpha$.

Note: Due to the discrete ~~natur~~ distribution the summed probability contains more than α, which is unavoidable and gives rise to conservatism.